

# Continuous Random Variables

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## Relevant Functions

- Probability density function (pdf) of r.v.  $X$ :  $f_X(x)$

$$P[a \leq X \leq b] = \int_a^b f_X(x) dx$$

- Cumulative distribution function (CDF):

$$F_X(x) = P[X \leq x]$$

- Tail of the distribution (reliability function):

$$R_X(x) = P[X > x] = 1 - F_X(x)$$

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## Moments

- k-th moment:  $E[X^k] = \int_{-\infty}^{+\infty} x^k f_X(x) dx$
- Expected value (mean): first moment

$$\mathbf{m} = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

- k-th central moment:

$$E[(X - \mathbf{m})^k] = \int_{-\infty}^{+\infty} (x - \mathbf{m})^k f_X(x) dx$$

- Variance: second central moment

$$\mathbf{s}^2 = E[(X - \mathbf{m})^2] = \int_{-\infty}^{+\infty} (x - \mathbf{m})^2 f_X(x) dx$$

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## The Uniform Distribution

- pdf:  $f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

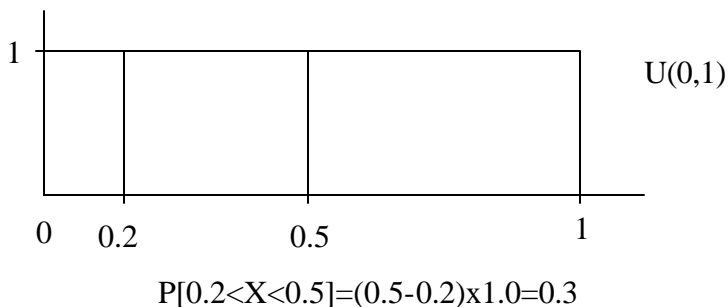
- Mean:  $\mathbf{m} = \frac{a+b}{2}$

- Variance:  $\mathbf{s}^2 = \frac{(b-a)^2}{12}$

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## The Uniform Distribution

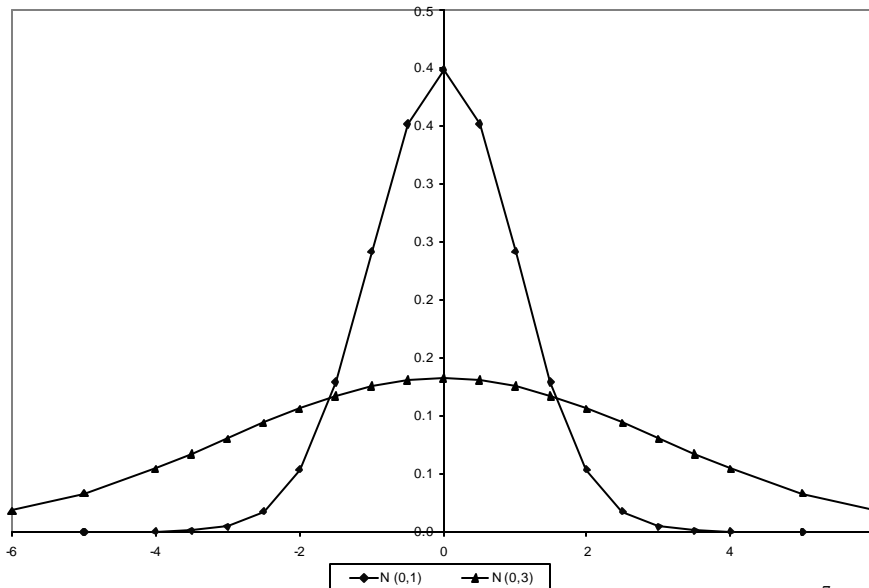


## The Normal Distribution $N(\boldsymbol{\mu}, \boldsymbol{\sigma})$

- Many natural phenomena follow a normal distribution.
- The normal distribution can be used to approximate the binomial and the Poisson distributions.
- Two parameters: mean and standard deviation.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\boldsymbol{\sigma}} e^{-(1/2)[(x-\boldsymbol{\mu})/\boldsymbol{\sigma}]^2}$$

## The Normal Distribution $N(\mu, \sigma)$



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## The Standard Normal Distribution

- To use tables for computing values related to the normal distribution, we need to standardize a normal r.v. as

$$Z = \frac{X - \mu}{\sigma}$$

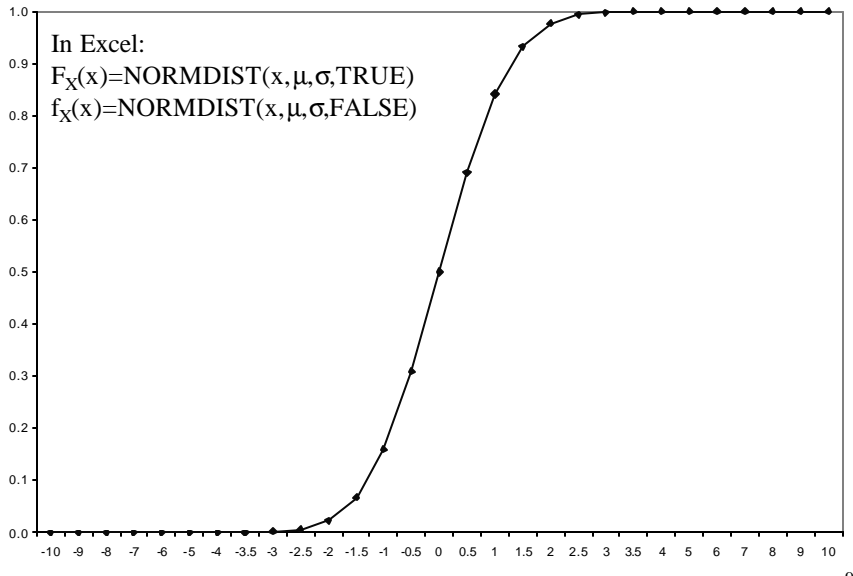
*standard normal score*

- Given  $X$ , compute a  $Z$  value  $z$ .
- Find the area value in a Table ( $\text{Prob } [0 < Z < z]$ ).

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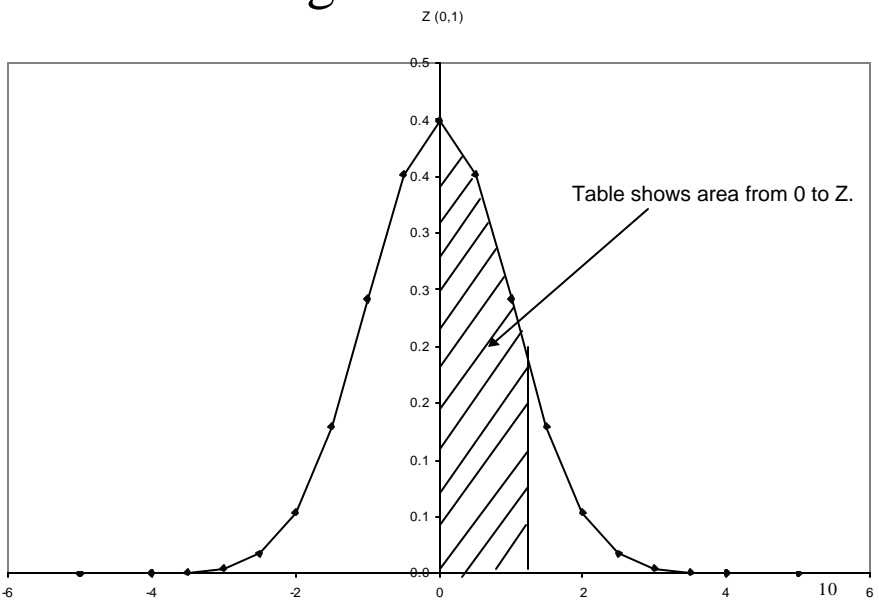
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# Normal CDF



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# Using Normal Tables



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## The Normal as an Approximation to the Binomial Distribution

- The normal can approximate the binomial if the variance of the binomial

$$np(1-p) \geq 10$$

- Binomial:  $m = np$

$$s = \sqrt{np(1-p)}$$

- Transformation:  $Z = \frac{X - np}{\sqrt{np(1-p)}}$

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## The Normal as an Approximation to the Binomial Distribution

- Consider a binomial r.v.  $X$  with average 50 and variance 25. What is  $P[50 \leq X \leq 60]$ ?

- Transformation:  $Z = \frac{X - 50}{\sqrt{25}} = \frac{60 - 50}{5} = 2.0$

- Using the table, the area between 50 and 60 for  $Z=2.0$  is 0.4772. So,

$$P[50 \leq X \leq 60] = 0.4772$$

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## The Normal as an Approximation to the Poisson Distribution

- The normal can approximate the Poisson distribution if  $\lambda > 5$ .

- Poisson: 
$$\begin{aligned} m &= I \\ s &= \sqrt{I} \end{aligned}$$

- Transformation: 
$$Z = \frac{X - I}{\sqrt{I}}$$

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## The Lognormal Distribution

- It is a random variable such that its natural logarithm has a normal distribution.

$$f_X(x) = \frac{1}{x\sqrt{2ps_{\ln X}}} e^{-(1/2)[(\ln x - m_{\ln X})/s_{\ln X}]^2} \quad x > 0$$

$$Y = \ln X \quad (X \text{ and } Y \text{ are r.v.'s}) \quad \text{and} \quad Y = N(\mu, \sigma)$$

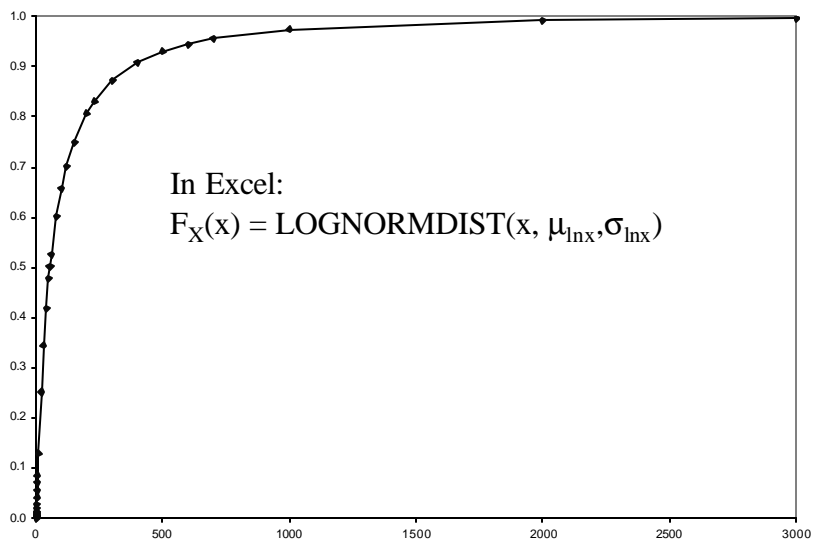
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# The Lognormal distribution

- Mean:  $E[X] = e^{m_{\ln X} + s_{\ln X}^2 / 2}$
- Standard Deviation:  $s = \sqrt{e^{2m_{\ln X} + s_{\ln X}^2} \cdot (e^{s_{\ln X}^2} - 1)}$

# Lognormal CDF



## The Exponential Distribution

- Widely used in queuing systems to model the inter-arrival time between requests to a system.
- If the inter-arrival times are exponentially distributed then the number of arrivals in an interval  $t$  has a Poisson distribution and vice-versa.

$$f_X(x) = \lambda e^{-\lambda x} \quad F_X(x) = 1 - e^{-\lambda x} \quad x \geq 0$$

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## The Exponential Distribution

- Mean and Standard Deviation:

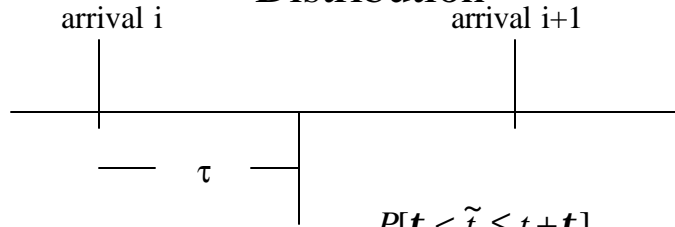
$$m = s = 1 / \lambda$$

- The COV is 1. The exponential is the only continuous r.v. with COV=1.
- The exponential distribution is “memoryless.” The distribution of the residual time until the next arrival is also exponential with the same mean as the original distribution.

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## Memoryless Property of the Exponential Distribution

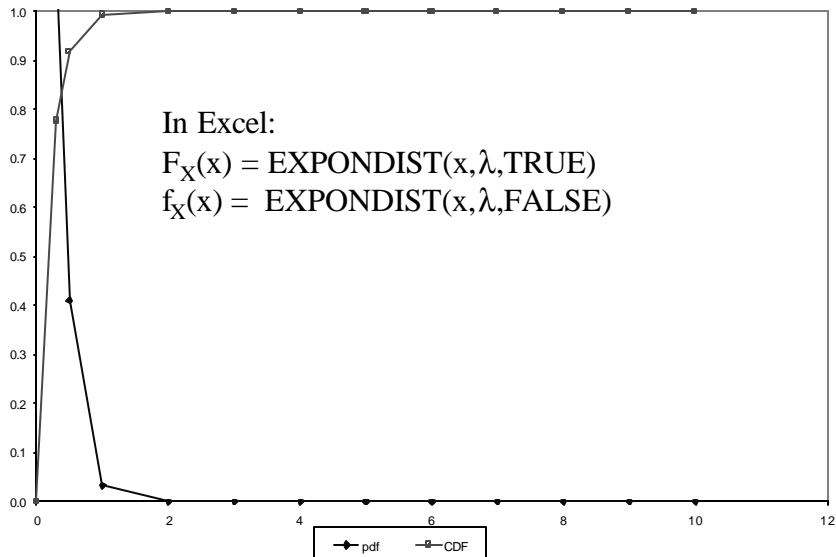


$$\begin{aligned}
 P[\tilde{t} \leq t+t \mid \tilde{t} > t] &= \frac{P[t < \tilde{t} \leq t+t]}{P[\tilde{t} > t]} \\
 &= \frac{P[\tilde{t} \leq t+t] - P[\tilde{t} \leq t]}{P[\tilde{t} > t]} \\
 &= \frac{1 - e^{-\lambda(t+t)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})} \\
 &= 1 - e^{-\lambda t}
 \end{aligned}$$

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## Exponential Distribution



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# Pareto Distribution

- A case of a heavy-tailed distribution.
- The probability of large values is not negligible.

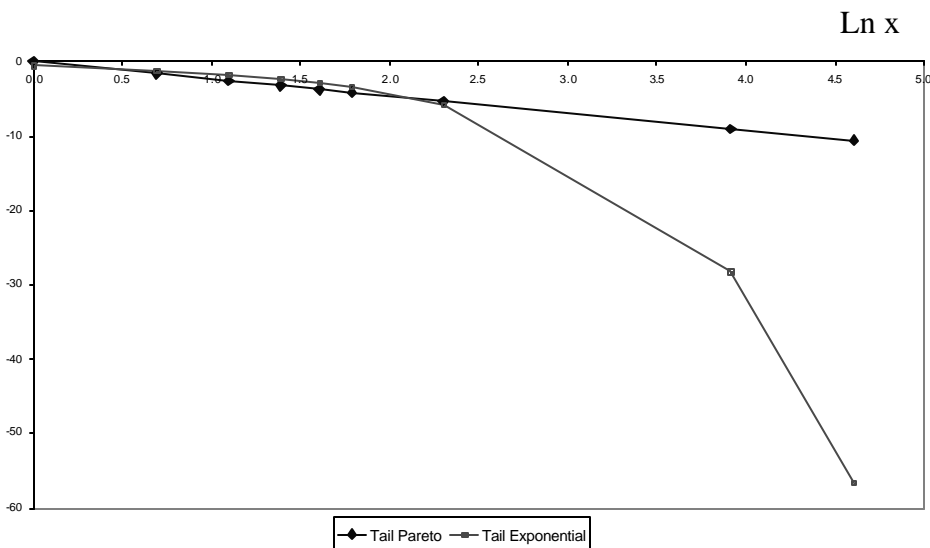
$$f_X(x) = \frac{a}{x^{1+a}} \quad a > 0, \quad x \geq 1$$

$$F_X(x) = 1 - \frac{1}{x^a} \quad a > 0, \quad x \geq 1$$

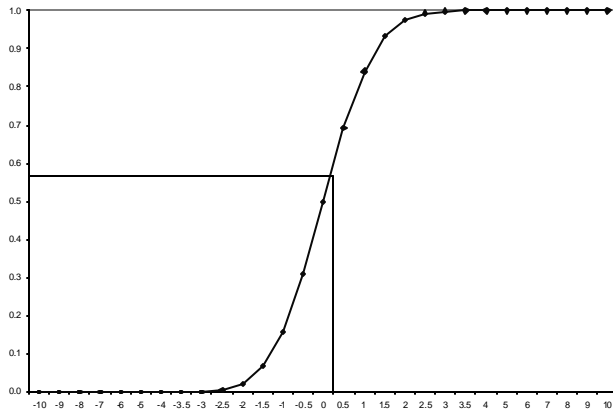
- Mean:  $\frac{a}{a-1} \quad a > 1$

- Variance:  $\frac{a}{(a-1)^2(a-2)} \quad a > 2$

Tail of the Pareto and Exponential Distributions



# Generation of Random Variables



- randomly generate a number  $u = U(0,1)$
- $x = F^{-1}(u)$  where  $F$  is the CDF

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