CS483-12 Transform-and-Conquer

Instructor: Fei Li

Room 443 ST II

Office hours: Tue. & Thur. 1:30pm - 2:30pm or by appointments

`lifei@cs.gmu.edu` with subject: CS483

`http://www.cs.gmu.edu/∼ lifei/teaching/cs483_fall07/`

Based on *Introduction to the Design and Analysis of Algorithms* by Anany Levitin and Professor Jyh-Ming Lien's notes.

Outline

- Transform and Conquer Techniques (which allow us to handle **dynamic** data / information)
  - Binary search tree
  - AVL tree via **rotations** (or red-black tree or splay tree)
  - $2-3$ tree (or $2-3-4$ tree or B tree)
  - Heap

Priority Queue

➢ Consider problems that require you to:

- schedule tasks (e.g., CPU)
- match n men to n women (eHarmony.com)
- route mails (Internet package routing)

➢ All these problems need to deal with dynamic data/information and contain information about priority/ordering/preference.

Priority Queue

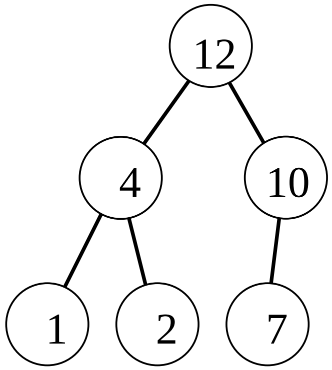
- Consider problems that require you to:
  - schedule tasks (e.g., CPU)
  - match n men to n women (eHarmony.com)
  - route mails (Internet package routing)
- All these problems need to deal with dynamic data/information and contain information about priority/ordering/preference.
- A priority queue is needed in these problems to perform the following operations:
  - Find the element with the highest priority
  - Delete the element with the highest priority
  - Insert element

Priority Queue

- Options for building a priority queue
  - a pointer points to the highest priority (what's the drawback?)
  - a sorted array (what's the drawback?)
  - a sorted list (what's the drawback?)
  - a balanced binary search tree (what's the drawback?)

Heap

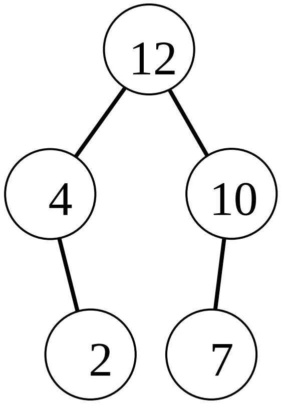
- Heap is a binary tree with keys at its nodes (one key per node) such that:
  - It is essentially complete, i.e., all its levels are full except possibly the last level, where only some rightmost keys may be missing
  - For each node n in a heap, n's key is always larger than the keys of n's kids (so, the largest value is in the root)



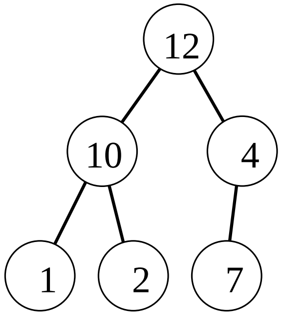
- Heap is data structure good for building priority queue.

Heap

- Only right most leaves are allowed empty (easier to expand, delete, and store). See the non-heap.



- Heap's elements are ordered top down (along any path down from its root), but they are not ordered left to right.

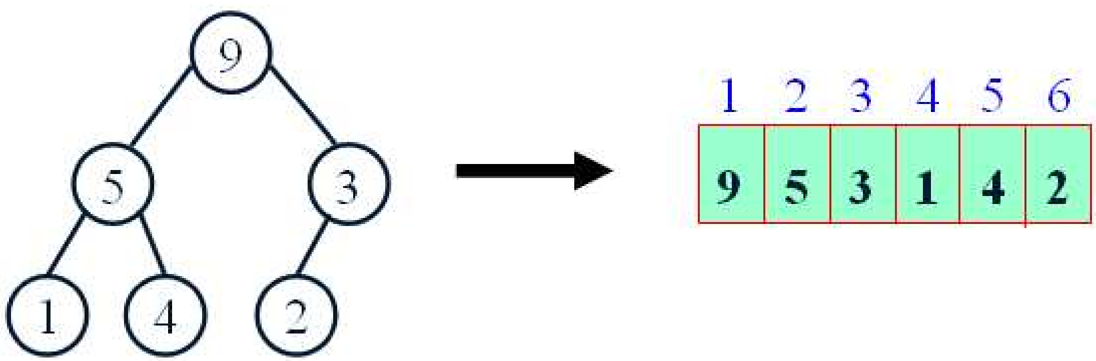


Properties of Heaps

- Given n, there exists a unique binary tree with n nodes that is essentially complete, with $h = \lceil \log_2 n \rceil$
- The root contains the largest key
- The subtree rooted at any node of a heap is also a heap
- A heap can be represented as an array

Heap's Array Representation

- Store heap's elements in an array (whose elements indexed, for convenience, 1 to n) in top-down left-to-right order
  - The kids of a node with index i have indices $2i$ and $2i+1$
  - The parent of a node with index i has index $\lfloor \frac{i}{2} \rfloor$
  - Parental nodes are represented in the first $n/2$ locations
- Example:



Heap: Insertion

- Assuming that we have a heap, and given a value with key k, insert the value to the heap.

Algorithm 0.1: HEAPINSERT(H, k)

Place k at $n+1$

Let $i = n+1$

while $H[i] > H[\lfloor \frac{n+1}{2} \rfloor]$ **and** $i > 0$

do $\begin{cases} \text{Swap } (H[i], H[\lfloor \frac{n+1}{2} \rfloor]) \\ i \leftarrow \lfloor \frac{n+1}{2} \rfloor \end{cases}$

- Time efficiency: $O(\log n)$
- Example:

Heap: Top-Down Construction

- Problem: Given an array $A[1 \cdots n]$ of orderable items, output a heap $H[1 \cdots n]$.
- A heap can be constructed by successive insertions of a new key into a previously constructed heap. That is, we can call HEAPINSERT iteratively over all the keys.

Algorithm 0.2: HEAPTOPDOWN($A[1 \cdots n]$)

$H \leftarrow A[1]$

for $i \in \{2 \cdots n\}$

  do HEAPINSERT$(H, A[i])$

Heap: Bottom-Up Construction

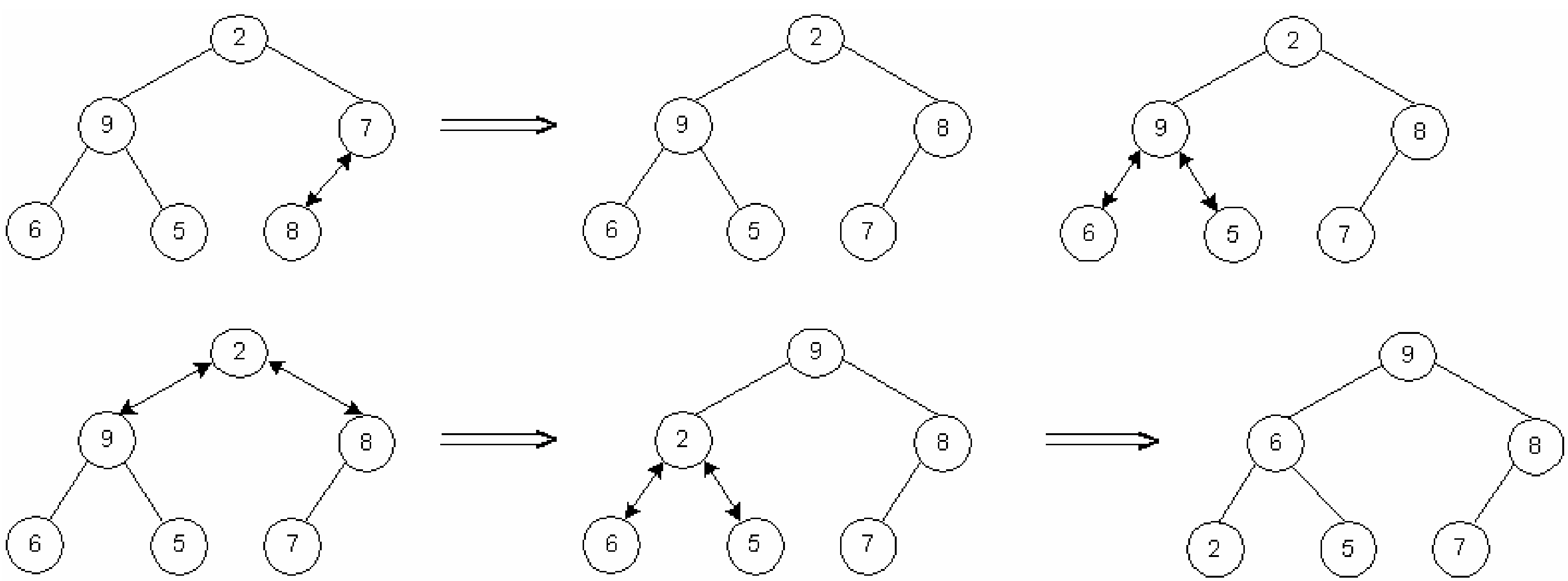
Step 0: Initialize the structure with keys in the order given

Step 1: Starting with the last (rightmost) parental node, fix the heap rooted at it, if it does not satisfy the heap condition: keep exchanging it with its largest child until the heap condition holds

Step 2: Repeat Step 1 for the preceding parental node

Heap: Bottom-Up Construction

Example: Construct a heap for the list $2, 9, 7, 6, 5, 8$



Heap: Bottom-Up Construction Algorithm

Algorithm 0.3: HEAPBOTTOMUP($A[1 \cdots n]$)

for $i \leftarrow \{\lfloor n/2 \rfloor \cdots 1\}$

do $k \leftarrow i; v \leftarrow A[k]$

heap $\leftarrow$ **false**

while **not** heap **and** $2 \cdot k \leq n$

do $j \leftarrow 2 \cdot k$

if $j < n$ there are two children

do if $A[j] < A[j+1]$

do $j \leftarrow j + 1$

if $v \geq A[j]$

do heap $\leftarrow$ **true**

else $A[k] \leftarrow A[j]; k \leftarrow j$

$A[k] \leftarrow v$

Heap: Deletion

➢ Deleting the maximum key from a heap is similar to inserting a key

Algorithm 0.4: HEAPDELMAX(H)

swap $(H[1], H[n])$

Let $i = 1$

while $(H[i] < H[2i]$ **or** $H[i] < H[2i+1])$ **and** $i < n$

$$\textbf{do} \begin{cases} \textbf{if } H[2i] > H[2i+1] \\ \quad \textbf{then} \begin{cases} \text{swap}(H[i], H[2i]) \\ i = 2i \end{cases} \\ \quad \textbf{else} \begin{cases} \text{swap}(H[i], H[2i+1]) \\ i = 2i+2 \end{cases} \end{cases}$$

➢ What's the time complexity?

➢ Example: Build a heap from this list: $\{2, 9, 7, 6, 5, 8\}$ and delete the root's key

Heapsort

1. Construct a heap for a given list of n keys
2. Repeat operation of root removal $n-1$ times:
   - Exchange keys in the root and in the last (rightmost) leaf
   - Decrease heap size by 1
   - If necessary, swap new root with larger child until the heap condition holds
3. In-space sorting

Stage 1 (heap construction)	Stage 2 (root/max removal)
$1\ 9\ 7\ 6\ 5\ 8$	$9\ 6\ 8\ 2\ 5\ 7$
$2\ 9\ 8\ 6\ 5\ 7$	$7\ 6\ 8\ 2\ 5 \mid 9$
$2\ 9\ 8\ 6\ 5\ 7$	$8\ 6\ 7\ 2\ 5 \mid 9$
$9\ 2\ 8\ 6\ 5\ 7$	$5\ 6\ 7\ 2 \mid 8\ 9$
$9\ 6\ 8\ 2\ 5\ 7$	$7\ 6\ 5\ 2 \mid 8\ 9$
	$2\ 6\ 5 \mid 7\ 8\ 9$
	$6\ 2\ 5 \mid 7\ 8\ 9$
	$5\ 2 \mid 6\ 7\ 8\ 9$
	$5\ 2 \mid 6\ 7\ 8\ 9$
	$2 \mid 5\ 6\ 7\ 8\ 9$

Heapsort

- Pop the largest element from the heap, i.e., call HEAPDELMAX $(n-1)$ times
- What's the complexity?

  $O(n \log n)$