CS 310: Heapify and HeapSort

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Week 14-2

Binary Heaps

Great for building Priority Queues:

Op	Worst Case	Avg Case
`findMin()`	$O(1)$	$O(1)$
`insert(x)`	$O(\log N)$	$O(1)$
`deleteMin()`	$O(\log N)$	$O(\log N)$

Questions

- What can't one do with a binary heap that is possible with BSTs and hash tables?
- What else can one do with binary heaps aside from building priority queues?

Exercise: Sort data using a Binary Heaps/PQ

- Sort an array of stuff with a PQ/Binary heap
- Define the following method

```
// Sort an array using a priority queue
public static <T extends Comparable>
void heapSort(T data[]){ ... }
```

- `T` is `Comparable`
- `data[]` unsorted
- return: nothing, but `data` must be sorted after method finishes
- Use `pq.insert(x)` and `pq.deleteMin()`
- What is the complexity of `heapSort()`?
  - Runtime complexity?
  - Memory complexity?

Out of Place Sort Sucks

Initial solution required data duplication

- Copy from `data` to `pq`, then back
- Out of place sorting, double memory requirement

For large arrays this hurts

- Want truly in place sorting
- Don't make copy, avoid $O(N)$ space overhead

Is this possible for heap sort?

In-Place Sorting with Heaps: Three Problems

1. 1-indexing of heaps

- Not a problem: heap root at index 0
- Formulas for `left(i)`, `right(i)`, `parent(i)`?

2. Removing min

- Have a heap, repeatedly Remove the min
- Where to put it?

Exercise: How do you solve (1) and (2)?

3. Broken Heap

- Sorting methods start with unsorted array
- Heap property doesn't hold - how to fix it?

1. Formulas for Different Root Locations

Root at 1

```
static int root(){ return 1; }
static int left(int i){ return i*2; }
static int right(int i){ return i*2+1; }
static int parent(int i){ return i / 2; }
```

Root at 0

```
static int root(){ return 0; }
static int left(int i){ return i*2+1; }
static int right(int i){ return i*2+2; }
static int parent(int i){ return (i-1) / 2; }
```

2. In Place Heap Sort

If we have a heap already...

Space Available

- Remove an element from a heap
- Now open space at end of array (percolate down)
- Put the removed element at end of array
- Repeat until empty

Min and Max Heap

Above process orders the array

- For Min-Heap will result in biggest to smallest
- For Max-Heap will result in smallest to biggest

Problem 3: Unsorted array to Heap

- How does one go from an unsorted array to a heap ordered array?

Heapify, a.k.a. `buildHeap()`

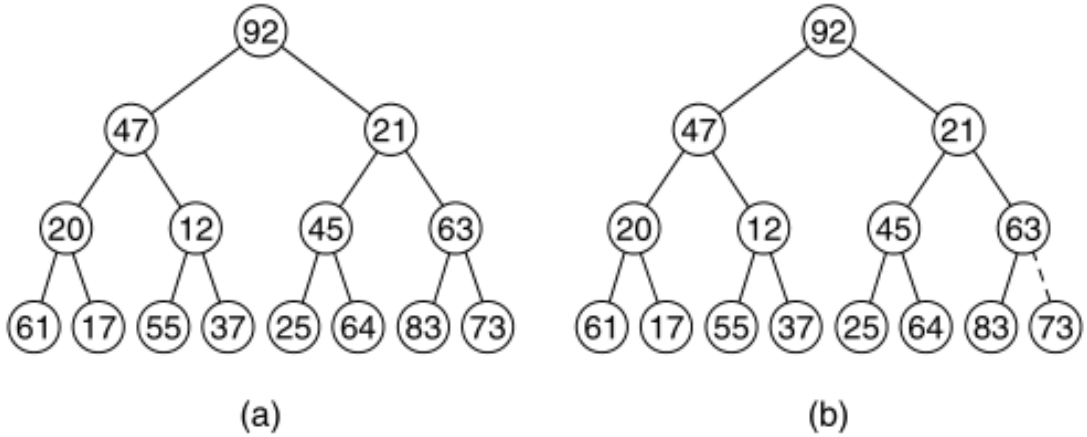
Converts an existing array into a heap (!)

```
public void buildHeap( ) {
  for( int i = parent(this.size); i >= root(); i-- ){
    this.percolateDown( i );
  }
}
```

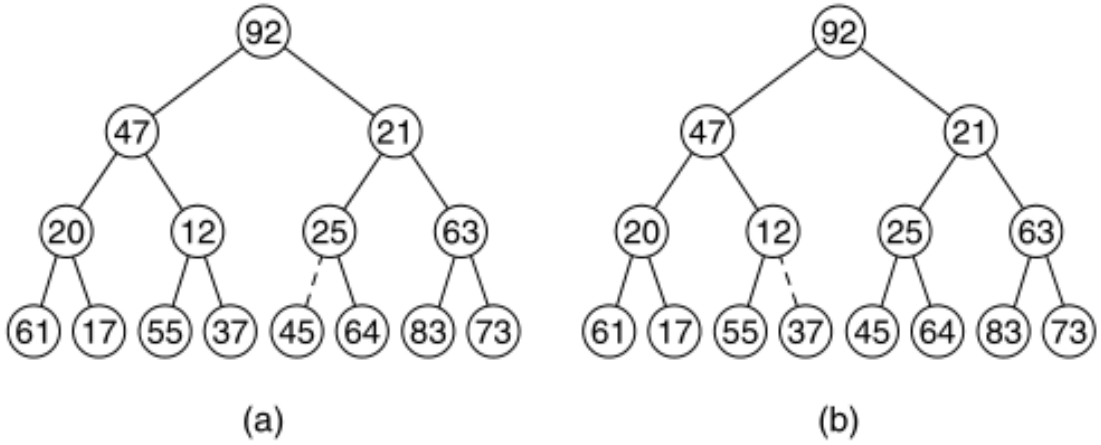
Build the heap bottom up, repeated `percolateDown(i)`

- Start one level above bottom (where for size N heap?)
- Work right to left, low to high
- If small guy is down low, will bubble up

Heapify Example

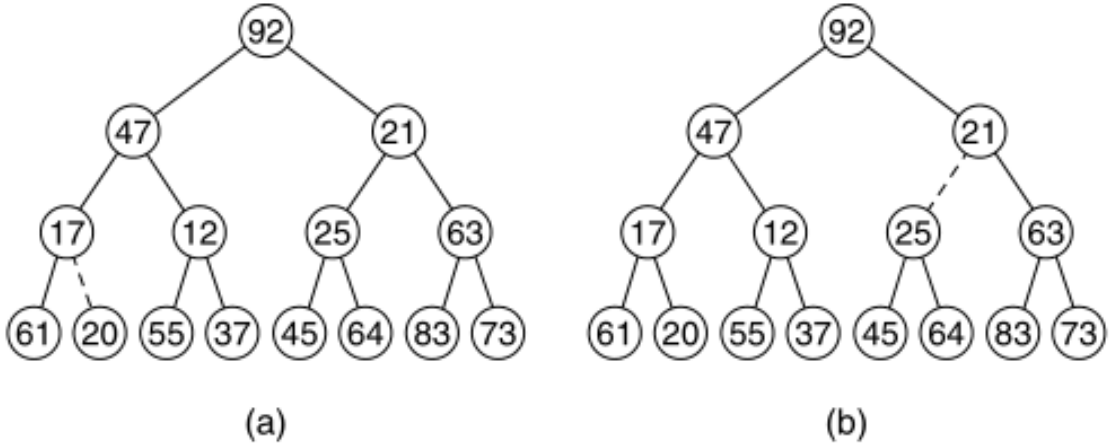


Level 3: Initial, `percolateDown([63])`

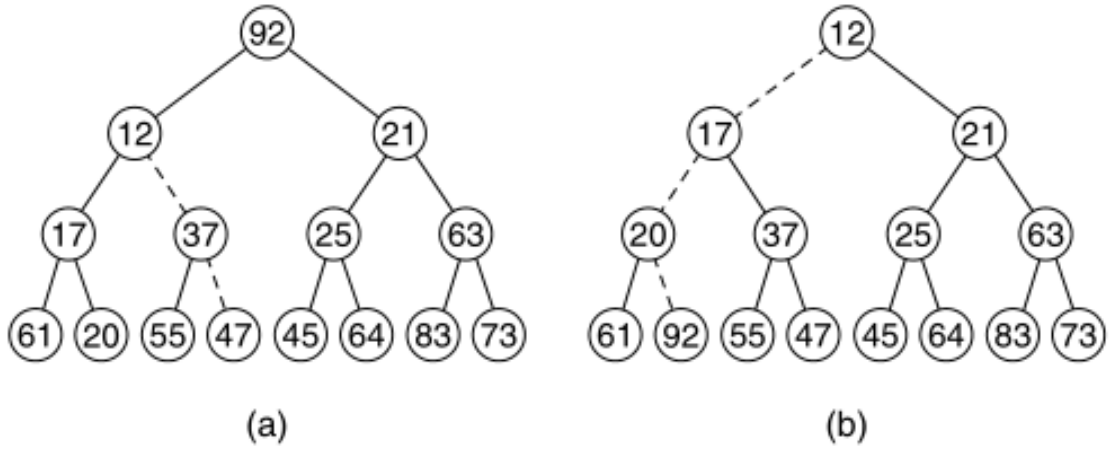


Level 3: `percolateDown([45])`, `percolateDown([12])`,

Heapify Example



Level 3: `percolateDown([20])`, Level 2: `percolateDown([21])`,



Level 2: `percolateDown([47])`, Level 1: `percolateDown([92])`,

Exercise: Build me a Heap as Quick as You Can

```
public void buildHeap( ) {
  for( int i = parent(this.size); i >= root(); i-- ){
    this.percolateDown( i );
  }
}
```

Discuss with a neighbor

- What is the runtime complexity of Heapify / `buildHeap()`?
- How many small moves versus big moves?
- Derive an expression for the worst possible number of moves

Complexity of Heapify

Heap size n, height h, assume complete (?)

Measure *level* from bottom

- Level 1 is bottom, has $2^{\text{h-1}}$ nodes,
- Level 2 is second from bottom, has $2^{\text{h-2}}$ nodes
- Level i is *ith* form bottom, has $2^{\text{h-i}}$ nodes
- Level h is root, has $2^{\text{h-h}} = 1$ node

Each level i node can move i down so

$$\begin{aligned} \mathit{moves} &= \sum_{i=1}^{h} i \times 2^{h-i} = \sum_{i=1}^{\log_2 n} i \times 2^{\log_2 n - i} \\ &= \sum_{i=1}^{\log_2 n} i \times \frac{2^{\log_2 n}}{2^i} = n \sum_{i=1}^{\log_2 n} \frac{i}{2^i} \\ &\leq n \times 2 = O(n) \end{aligned}$$

because

$$\sum_{i=1}^{\infty} \frac{i}{2^i} \to 2$$

Summary of Heap Sort

```
Input: Array a
Output: a is sorted

Build Max Heap on a
for i=0 to length-1 a
  tmp = findMax(a)
  removeMax(a)
  a[length-i-1] = tmp
done
```