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Security:

Challenger | Adv

$s \leftarrow \mathrm{Gen}(1^n)$

$\xrightarrow{\quad s \quad}$

$\xleftarrow{\quad x,y \quad}$

$H^s(x) = H^s(y)$

$x \neq y$

Unkeyed Hash Functions in practice

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In practice, we don't know of any adversaries that can find any collision in SHA-256 or SHA-3, so we use these unkeyed hash functions anyway.

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Exercise: prove that if (Gen, h) is collision resistant, then $(\mathrm{Gen}, \hat{h})$ is collision resistant.
Clearly the output of $\hat{h}$ is not random looking!

Weaker Security Notions

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Second Preimage Resistance:
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$\mathcal{A}$ has to output $\hat{x}$ such that $h^s(\hat{x}) = h^s(x)$.

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$\hat{\mathcal{A}}$ can simply sample x itself, run $\mathcal{A}$ to get $\hat{x}$, and output collision $x, \hat{x}$.

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How likely is it that $\hat{x} = x$?
Note that $\mathcal{A}$ is only given y and does not know how it was computed:
x was sampled at random, $y = h^s(x)$.
Just as easily, it could have been $\hat{x}$ sampled at random, and $y = h^s(\hat{x})$!

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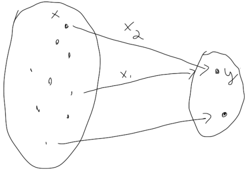

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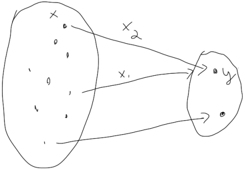

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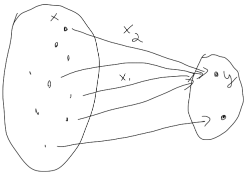

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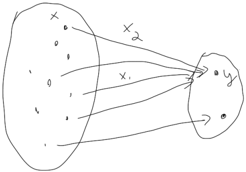

$$\Pr[\hat{x} \neq x] = \frac{|h^{-1}(y)|-1}{|h^{-1}(y)|}$$