CS689 - Robot Motion Planning

Motion Planning with Kinematics and Dynamics

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Beyond Motion Planning with Geometric Constraints

- Geometric constraints are generally not sufficient to adequately express robot motions

[movie: Moving Car 1]

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- Are the motions realistic?
- What is missing?

Motion Planning with Kinematic Constraints

- Actual car steering and constraints on velocity may make planned motions more realistic.

[movie: Moving Car 2]

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[movie: Moving Car 2]

- Are the motions more realistic?
- Can they be made more realistic?
- How?

Motion Planning with Kinematic and Dynamic Constraints

- Geometric constraints are generally not sufficient to adequately express robot motions
- Constraints on velocity, forces, torques, accelerations are needed for better representations

[movie: Moving Car 1 - Geometric]

[movie: Moving Car 1 - Kinematics]

[movie: Moving Car 3 - Dynamics]

Kinematics Constraints == Constraints on Velocity

Illustration:

- C-space $= \mathbb{R}^2 = \{q = (x, y) \in \mathbb{R}^2\}$
- Velocity $\frac{dq}{dt} = \dot{q} = (\dot{x}, \dot{y})$
- Each $(\dot{x}, \dot{y})$ is an element of the tangent space $T_q(\mathbb{R}^2)$, which is a 2D vector space at every (x, y)
- At each $q \in \mathbb{R}^2$, restricting the set of velocities yields some set $U(q) \subset T_q(\mathbb{R}^2)$
- Think about the kinds of constraints imposed on velocity

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- $\dot{x} + \dot{y} \geq 1$: Impossible to stop or slow down.

Implicit vs. Parametric Kinematics Constraints

Implicit and parametric representations are alternative ways to express

$$U(q) \quad \forall q \in \mathbb{R}^2.$$

- Implicit (indirect) representation: expresses velocities that are not allowed.
- Parametric (direct) representation: expresses velocities that are allowed.

Implicit Velocity Constraints

Implicit velocity constraints express velocities that are <u>not allowed</u> and are of the form:

$$g(q, \dot{q}) \bowtie 0$$

where

- $g(q, \dot{q})$ is some function $g : Q \times \dot{Q} \rightarrow \mathbb{R}$
- $\bowtie$ can be any of the symbols $=, <, >, \leq, \geq$

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Examples of implicit velocity constraints

- $\dot{x} > 0$, $\dot{x} = 0$, $\dot{x}^2 + \dot{y}^2 \leq 1$, $x = \dot{x}$, etc.

Parametric Velocity Constraints

Parametric velocity constraints express velocities that are allowed and are of the form:

$$\dot{q} = f(q, u)$$

where

- $f(q, u)$ is some function $f : Q \times U \rightarrow \dot{Q}$ that expresses a set of differential equations.

  f is referred to as the configuration transition equation
- u is an input control/action.
- So, $T_q(Q)$ is *parameterized* through u: Given a (sampled?) control/action, one can obtain an *allowed* velocity.

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Let's work out the *kinematics* of some simple *wheeled* systems.

Kinematics for Wheeled Systems

- Objective 1: Derive configuration transition equation (*do the kinematics*) for wheeled systems (car, differential drive, and unicycle).

  Constrain velocities for more realistic motions.

- Objective 2: Proceed with dynamics after working out kinematics.

  Constrain accelerations for even more realistic motions.

a simple car as opposed to other car variations

Objective: Obtain f as in $\dot{q} = f(q, u)$.

Preliminaries:

- Car cannot drive sideways because

  .

- Parallel parking would be trivial

- Complicated maneuvers arise due to rolling constraints.

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Kinematics for Wheeled Systems – A Simple Car

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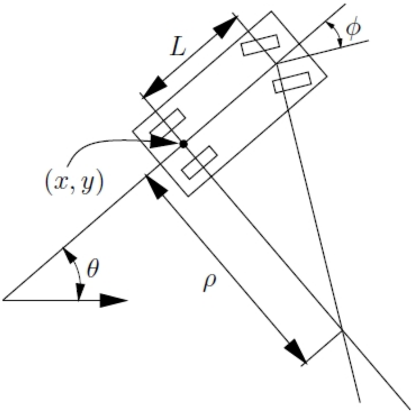
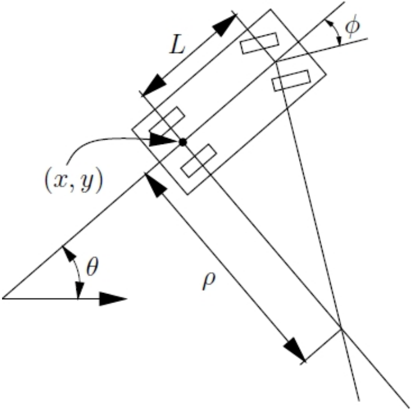
- Need: model for car
- Need: understand way car moves (what do we control?)



- Car: rigid body that moves in plane.
- Car configuration:
  $q = (x, y, \theta) \in \mathbb{R} \times S^1$
- Body frame:
  - Origin is at the center of rear axle
  - x-axis points along main axis of the car
- Velocity (signed speed in x direction of body frame): s
- Steering angle: ϕ
- Distance between front and rear axles: L

Kinematics for Wheeled Systems – A Simple Car

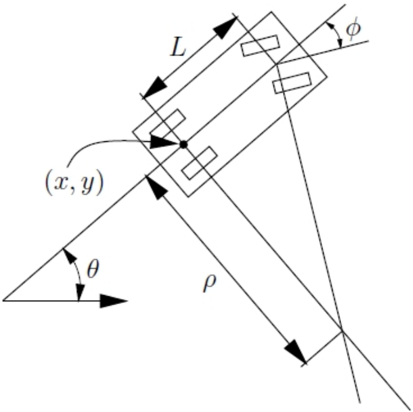


How does the car move?

- If steering angle ϕ is kept fixed, car travels in circular motion.
- Center of circle: intersection between normals to steering axis and car axis.
- Radius of circle: ρ

Need : Express car motions as a set of differential equations

- $\dot{x} = f_1(q, u)$
- $\dot{y} = f_2(q, u)$
- $\dot{\theta} = f_3(q, u)$



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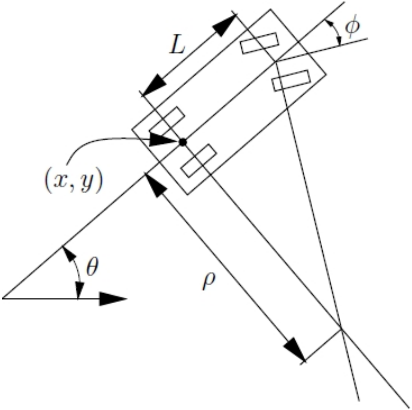

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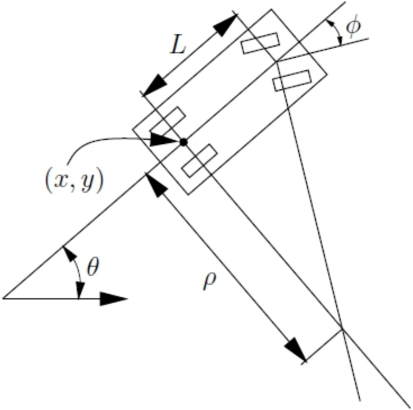

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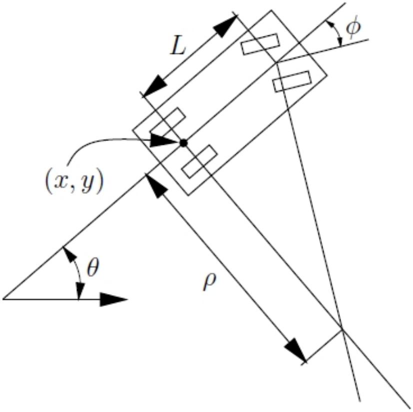

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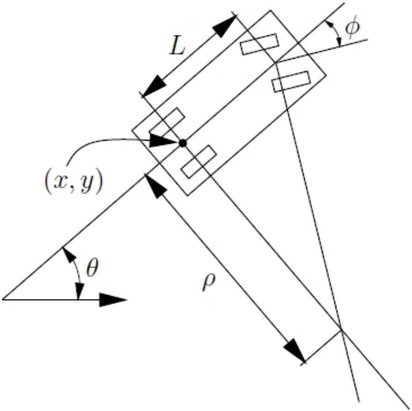

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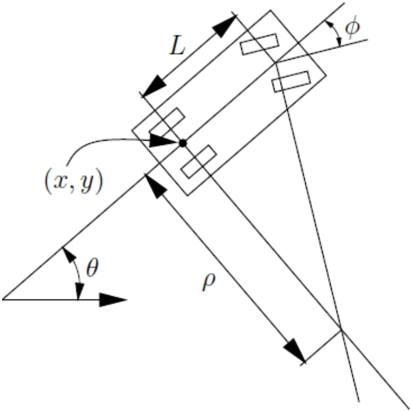

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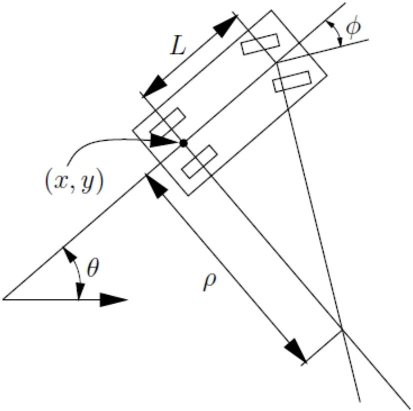

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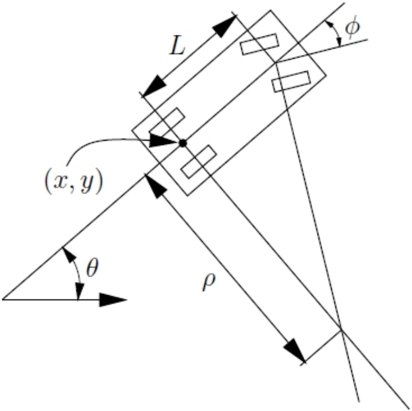

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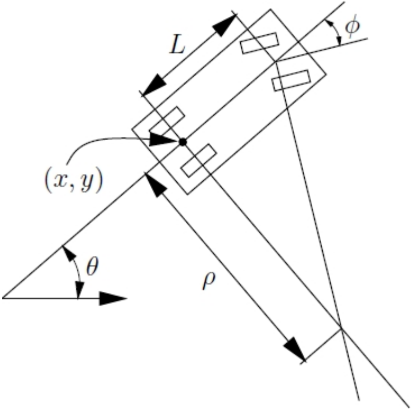
- What about $\dot{\theta}$?



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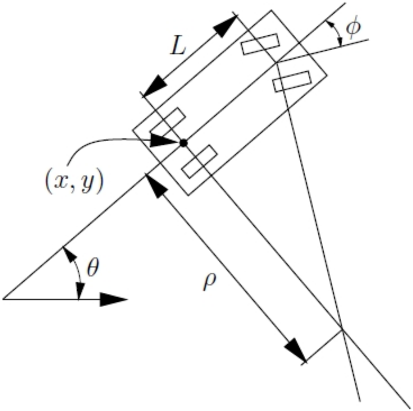

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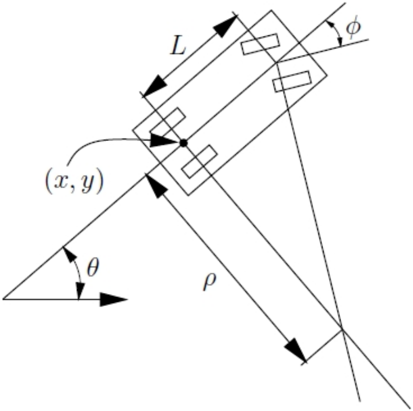

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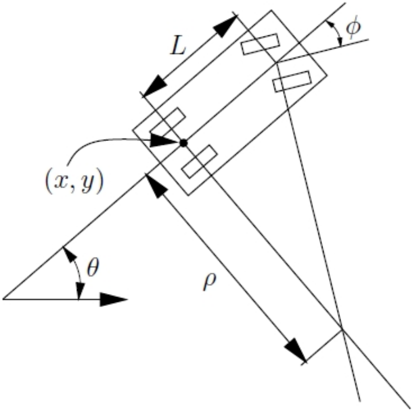

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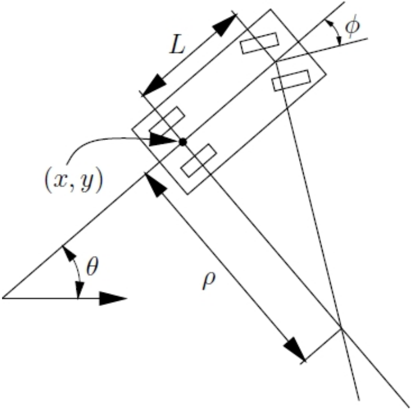

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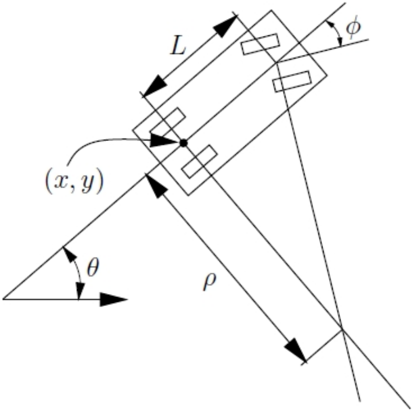

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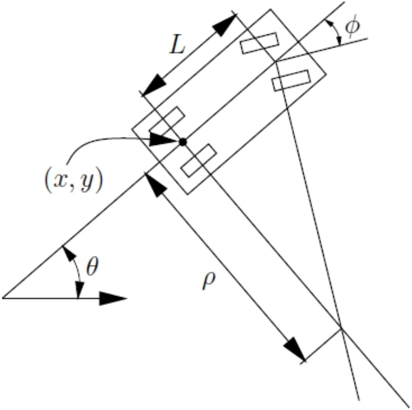

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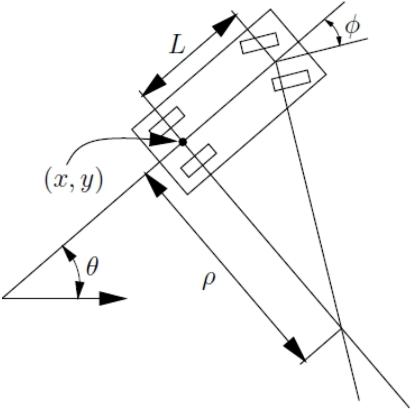

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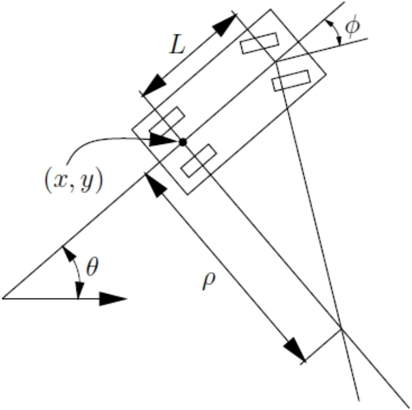
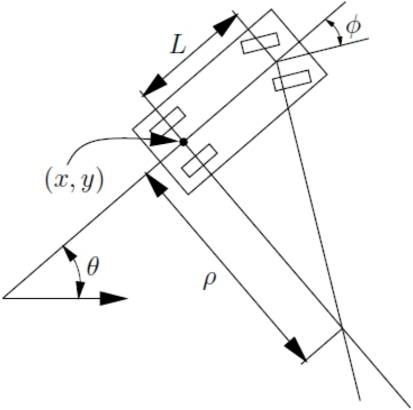
So, putting it all together:

$$d\theta = \frac{\tan\phi}{L} dw = \frac{\tan\phi}{L} s \Rightarrow \dot{\theta} = \frac{s}{L} \tan\phi$$



What we have so far on configuration transition equations:

Kinematics for Wheeled Systems – A Simple Car



What we have so far on configuration transition equations:

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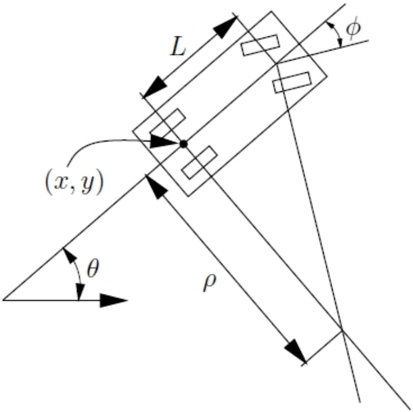

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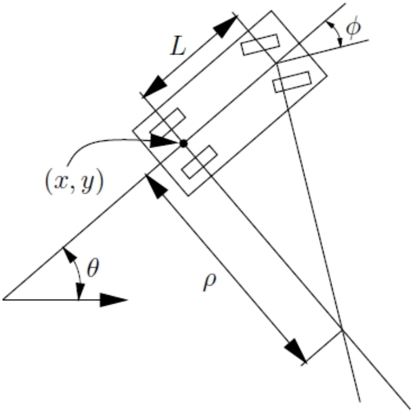
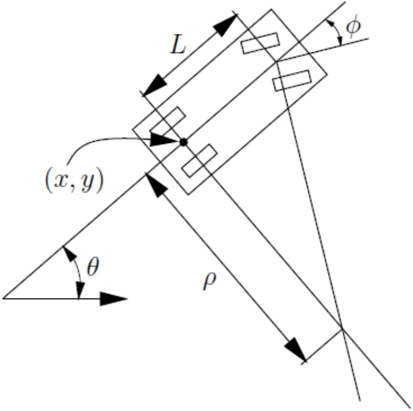
How should we control the car? Where are our controls/actions?



How should we control the car?



How should we control the car?

- Setting the speed s, i.e., $u_s = s$
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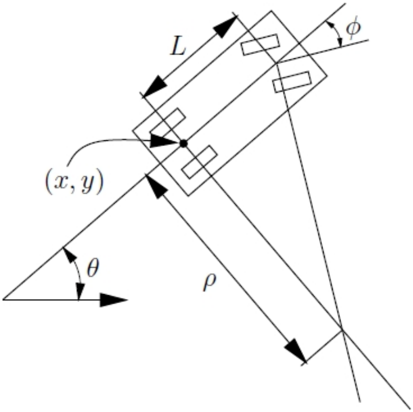

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Putting it all together:

- Input controls: u_s (speed) and u_ϕ (steering angle)
- CTE:
  $\dot{x} = u_s \cos\theta$
  $\dot{y} = u_s \sin\theta$
  $\dot{\theta} = \frac{u_s}{L} \tan u_\phi$

Forward Kinematics for Simple Car

Problem formulation when only worrying about geometric constraints:

- Standing at configuration $q = (x, y, \theta)$, what are the workspace coordinates of the control points?

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New problem formulation under kinematic constraints:

- Standing at configuration $q = (x, y, \theta)$ at time t, determine configuration $q^{'} = (x^{'}, y^{'}, \theta^{'})$ at time $t + \delta t$ given controls u_s, u_ϕ
- Approach: use CTE to obtain $\dot{q}$
- Update $q^{'} = q + \delta t \cdot \dot{q}$ [movie: Traj]

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How can I generate a random path in configuration space?

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- Standing at configuration $q = (x, y, \theta)$ at time t, determine configuration $q^{'} = (x^{'}, y^{'}, \theta^{'})$ at time $t + \delta t$ given controls u_s, u_ϕ
- Approach: use CTE to obtain $\dot{q}$
- Update $q^{'} = q + \delta t \cdot \dot{q}$ [movie: Traj]

How can I generate a random path in configuration space?
Sample values for controls.

Forward Kinematics for Simple Car

Problem formulation when only worrying about geometric constraints:

- Standing at configuration $q = (x, y, \theta)$, what are the workspace coordinates of the control points?
- Approach: treat car as rigid-body in 2D workspace, use combination of rotation by θ followed by translation by center point (x, y).

New problem formulation under kinematic constraints:

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How can I generate a random path in configuration space?
Sample values for controls. Should there be bounds?

Variations of the Simple Car Model

- Input controls: u_s (speed) and u_ϕ (steering angle)
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Dubins car

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- We have issue of constrained mobility (no instantaneous controls).

Inverse Kinematics for Simple Car

Problem formulation:

- Standing at configuration $q_{\text{start}} = (x_{\text{start}}, y_{\text{start}}, \theta_{\text{start}})$ at time t find path that places robot at $q_{\text{goal}} = (x_{\text{goal}}, y_{\text{goal}}, \theta_{\text{goal}})$ at time $t + \delta t$.

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New problem formulation: Find series of controls to get to goal.

- Motion planning with kinematic constraints to find *feasible* series of maneuvers.

Constrained Mobility in Non-holonomic Systems

- Simple Car is under-actuated: only 2 controls, but C-space has 3 dimensions.

A robot is non-holonomic if its motion is constrained by a non-integrable equation of the form $f(q, \dot{q}) = 0$.

- Simple Car is non-holonomic because $-\dot{x}sin\theta + \dot{y}cos\theta = 0$.
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  Try to parallel park with no reverse gear!

  Can do it in an infinitely-large parking lot with no obstacles.

Kinematics for Other Wheeled Systems

Other non-holonomic wheeled systems:

- Differential drive
- Unicycle
- Tractor trailer

Objective 1: Derive CTE for each of them

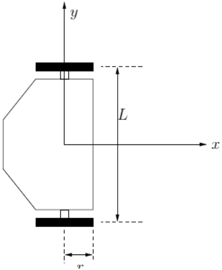
Objective 2: Move on to dynamic constraints

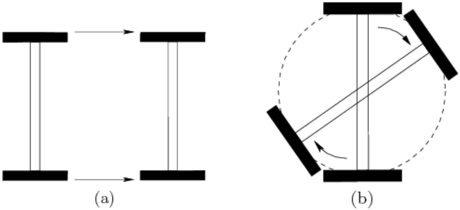
Put it all together for sampling-based motion planning.

Kinematics for Wheeled Systems – Differential Drive

Differential drives

- Most indoor robots are modeled after ddrives.
- Two main wheels, each attached to its own motor.
- Third invisible (caster) wheel in rear to passively roll and prevent falling over.
- Wheels move at same or different angular velocity.
- As a result, ddrive moves ahead or on circle.





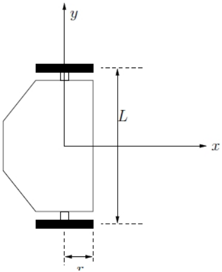
Body frame:

- Origin at center of axle
- x-axis perpendicular to axle
- L: distance between wheels.
- r: wheel radius

(a) Pure translation when both wheels move at same angular velocity

(b) pure rotation when wheels move at opposite velocities.

That is why origin placed at center of axle, so ddrive rotates in place in (b).



Input controls $v = (v_\ell, v_r)$

- v_ℓ: angular velocity of left wheel
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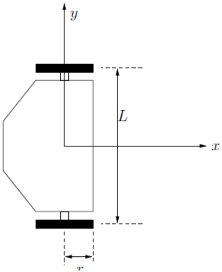

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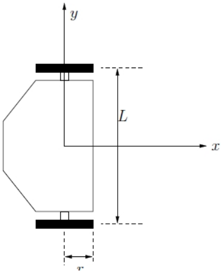

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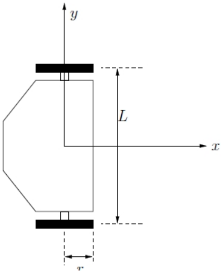
Kinematics for Wheeled Systems – Differential Drive



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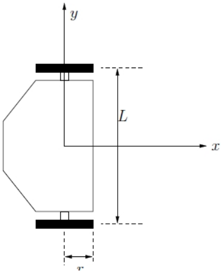

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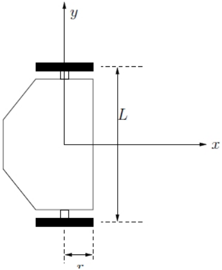
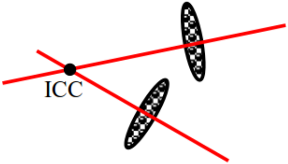
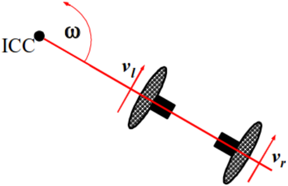
- CTE for ddrive?
- Variant of car, but need to introduce concept of ICC



There must be a point that lies along common left and right wheel axis, known as ICC – Instantaneous Center of Curvature.



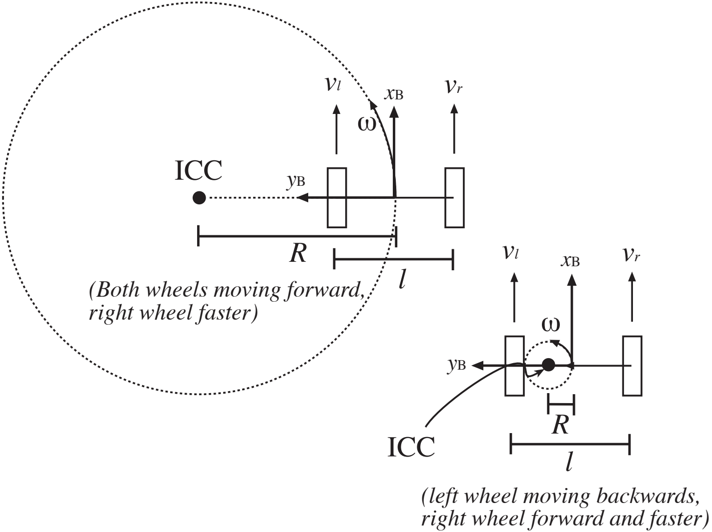


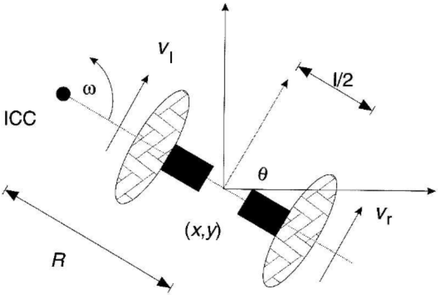
Ddrive is variant of simple car:

- $\dot{x} = s(v_\ell, v_r)\cos\theta$
- $\dot{y} = s(v_\ell, v_r)\sin\theta$
- $\dot{\theta} = f(v_\ell, v_r)$

For ddrive, ICC exists as long as $v_\ell \neq v_r$. Ddrive rotates around ICC.

ICC location on wheel axis changes as v_ℓ and v_r change.





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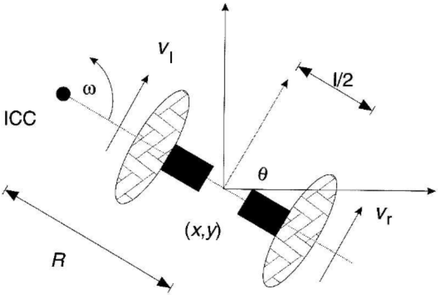

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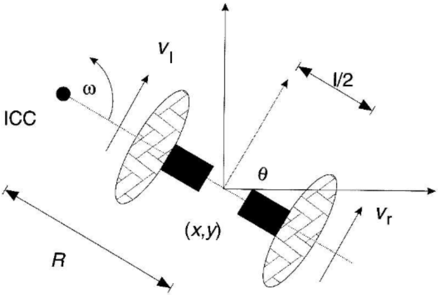

Observations to obtain CTE:

- Ddrive rotates around ICC
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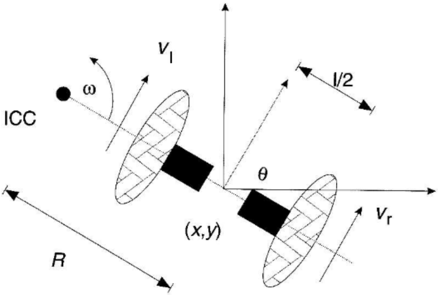

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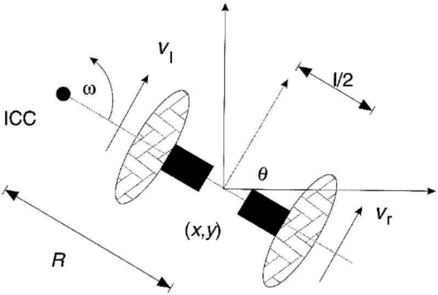
Arclengths traveled per unit of time by left and right wheel:



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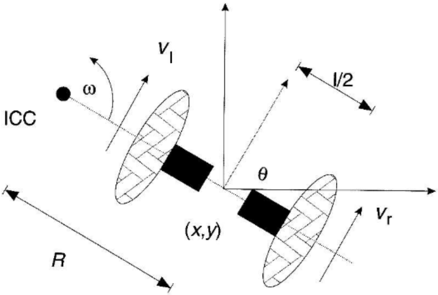
Kinematics for Wheeled Systems – Differential Drive



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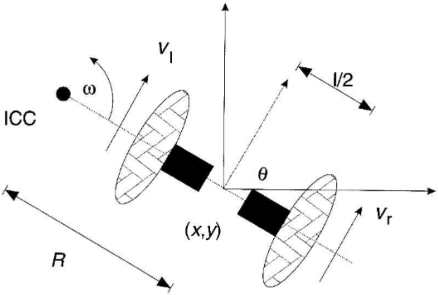

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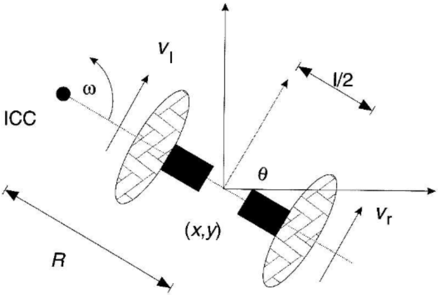

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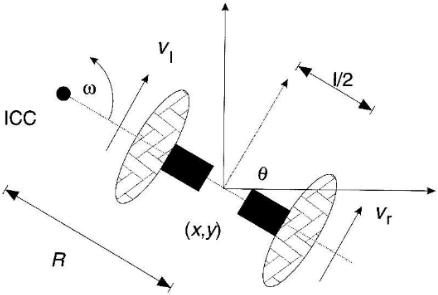

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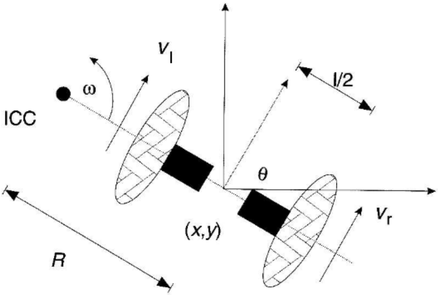

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Solving for w and R, we obtain:

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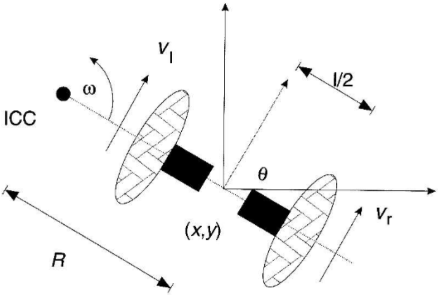

Ddrive is variant of simple car:

- $\dot{x} = s(v_\ell, v_r)\cos\theta$
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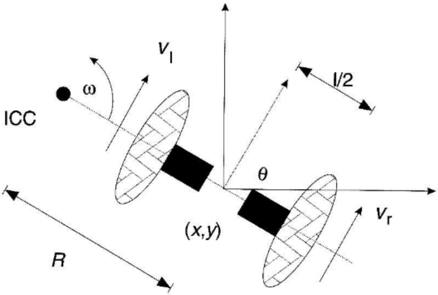
Arclengths traveled per unit of time by left and right wheel:

- $w \cdot (R - L/2) = v_\ell \cdot r$
- $w \cdot (R + L/2) = v_r \cdot r$

Solving for w and R, we obtain:

- $w = r/L \cdot (v_r - v_\ell)$

Kinematics for Wheeled Systems – Differential Drive



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Question: how does ω relate to $\dot{\theta}$?

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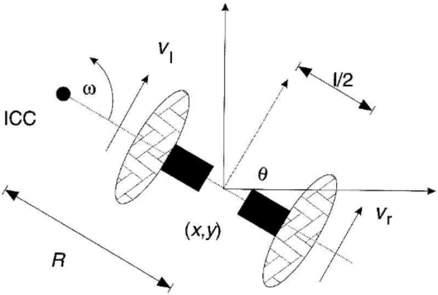

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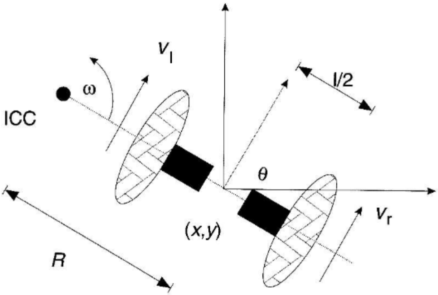
Answer: They are one and the same.



So:

- $\dot{\theta} = r/L \cdot (v_r - v_\ell)$
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s is translational velocity

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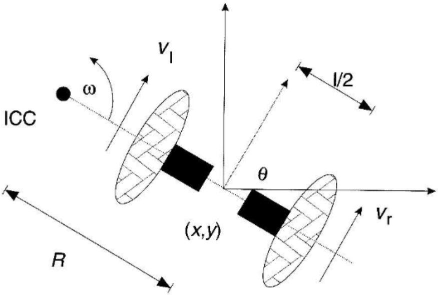

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So:

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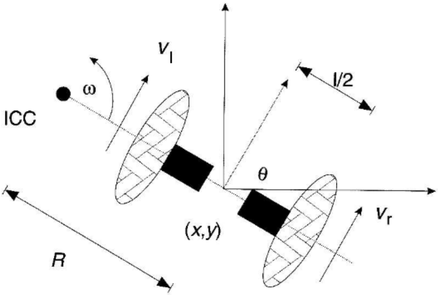

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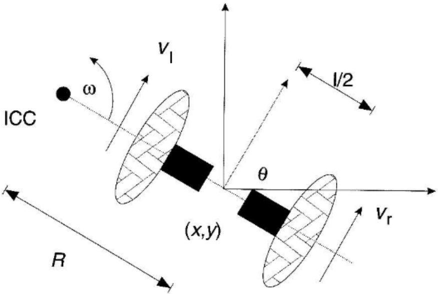


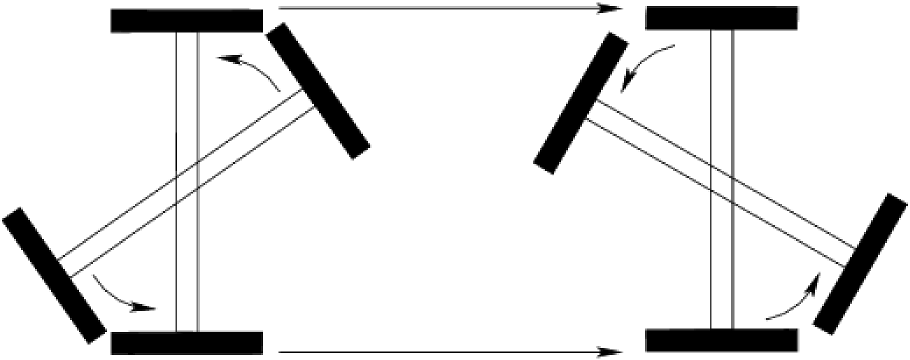
CTE (using u_ℓ and u_r for controls on wheel (angular) velocities:

- $\dot{x} = \frac{r}{2}(u_\ell + u_r)\cos\theta$
- $\dot{y} = \frac{r}{2}(u_\ell + u_r)\sin\theta$
- $\dot{\theta} = \frac{r}{L}(u_r - u_\ell)$

Interesting questions:

- What happens when either u_ℓ or u_r (not both) are set to 0?
- Can ddrive simulate motions of simple car?
- Is ddrive non-holonomic?
- Is it STLC?

Can ddrive move between any two configurations?



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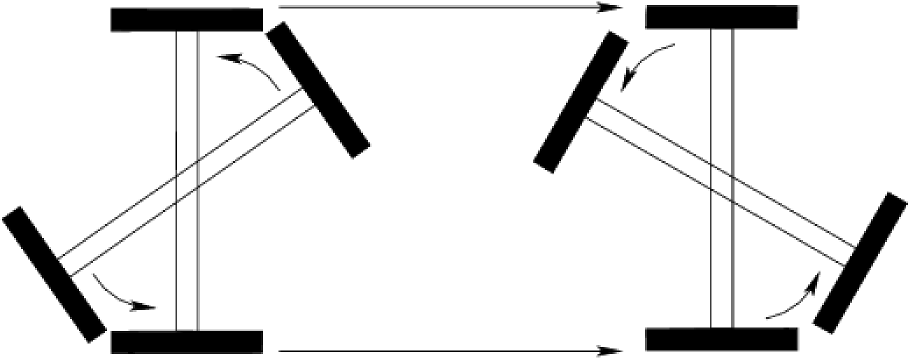
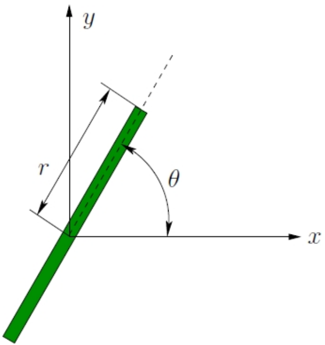
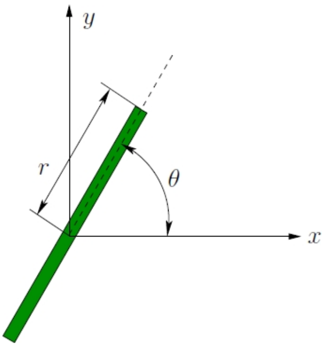


Point wheels as in destination. Translate. Rotate to desired orientation.

Kinematics for Wheeled Systems – Unicycle



- Rider can set pedaling speed and orientation of the wheel with respect to the z-axis
- r: wheel radius
- σ: pedaling angular velocity
- $s = r\sigma$: speed of unicycle
- ω: rotational velocity in the xy plane controlled directly.



CTE:

- $\dot{x} = u_\sigma r \cos\theta$
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- $\dot{\theta} = u_\omega$

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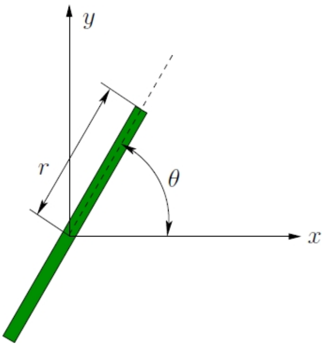

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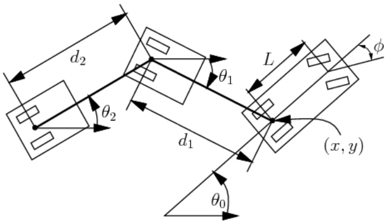
- Note: ddrive with $L = 1$, $u_s = ru_\sigma$
- Ddrive can simulate a unicycle. Unicycle can simulate simple car. Unicycle == tricycle.

Tractor Trailer



CTE:

- $\dot{x} = s \cos \theta$
- $\dot{y} = s \sin \theta$
- $\dot{\theta}_0 = s/L \tan \phi$
- $\dot{\theta}_1 = s/d_1 \sin(\theta_1 - \theta_0)$
- ...
- $\dot{\theta}_i = s/d_j(\Pi_{j=1}^{i-1} \cos(\theta_{j-1} - \theta_j)) \sin(\theta_{i-1} - \theta_i)$

- Simple car pulling k trailers, each attached to rear axle of body in front of it.
- New: hitch length, d_i, distance from center of rear axle of trailer i to point at which trailer is hitched to next body.
- Car itself contributes $\mathbb{R}^2 \times S^1$ to C, and each trailer contributes an S^1. So, $|\mathcal{C}| = k + 1$.
- Configuration transition equation is hard to get right. Shown one here is adapted from Murray, Sastry, IEEE Trans Autom Control, 1993.

[movie: strailer4]

Beyond Kinematics: Dynamical Systems

- Involve acceleration $\ddot{q}$ in addition to velocity $\dot{q}$ and configuration q
- Control acceleration directly
- Implicit constraints:
$$g(\ddot{q}, \dot{q}, q) = 0$$
- Parametric constraints:
$$\ddot{q} = f(\dot{q}, q, u)$$

State Space: Reducing Degree by Increasing Dimension

Example: $y \in \mathbb{R}$ is a scalar variable and

$$\ddot{y} - 3\dot{y} + y = 0 \qquad (1)$$

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Thus, (1) can be rewritten as two constraints

- $\dot{x}_1 = x_2$
- $\dot{x}_2 = 3x_2 - x_1$

Extending Models by Adding Integrators

Suppose equations of motions are given as:

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Procedure referred to as placing an integrator in front of u_i

Putting it all together: Car

Kinematic (first-order) model

Dynamics (second-order) model

Config $q = (x, y, \theta)$

- Position $(x, y) \in \mathbb{R}^2$
- Orientation $\theta \in S^1$

Control inputs $u = (u_s, u_\phi)$

- Signed speed $u_s \in \mathbb{R}$
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[movie: SCar]

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[movie: SDDrive]

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Putting it all together: Unicycle

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- Rotational velocity $u_\omega \in \mathbb{R}$

CTE $\dot{q} = f(q, u)$:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} u_\sigma r \cos\theta \\ u_\sigma r \sin\theta \\ u_\omega \end{bmatrix}$$

Dynamics (second-order) model

State $s = (x, y, \theta, \sigma, \omega)$

- Translational velocity $\sigma \in \mathbb{R}$
- Rotational velocity $\omega \in \mathbb{R}$

Control inputs $u = (u_1, u_2)$

- Translat. accel. $u_1 \in \mathbb{R}$
- Rotational accel. $u_2 \in \mathbb{R}$

CTE $\dot{s} = f(s, u)$:

$$\dot{s} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\sigma} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \sigma r \cos\theta \\ \sigma r \sin\theta \\ \omega \\ u_1 \\ u_2 \end{bmatrix}$$

Robot motions obtained by applying input controls and integrating equations of motions

Consider

- a starting state s_0
- an input control u
- motion equations $\dot{s} = f(s, u)$

Let $s(t)$ denote the state at time t. Then,

$$s(t) = s_0 + \int_{h=0}^{h=t} f(s(h), u)dh$$

Robot motions obtained by applying input controls and integrating equations of motions

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Computation can be carried out by

- Closed-form integration when available or
- Numerical integration

Numerical Integration – Euler Method

Let Δt denote a small time step. We would like to compute $s(\Delta t)$ as

$$s(\Delta t) = s(0) + \int_{h=0}^{h=\Delta t} f(s(h), u)dh$$

Euler Approximation

$$f(s(t), u) = \dot{s}(t) = \frac{ds(t)}{dt} \approx \frac{s(\Delta t) - s(0)}{\Delta t}$$

Therefore,

$$s(\Delta t) \approx s(0) + \Delta t\, f(s(t), u)$$

For example, Euler integration of the kinematic model of unicycle yields:

$$s(\Delta t) \approx \begin{bmatrix} x_0 \\ y_0 \\ \theta_0 \end{bmatrix} + \Delta t \begin{bmatrix} u_\sigma r \cos\theta \\ u_\sigma r \sin\theta \\ u_\omega \end{bmatrix}$$

- Advantage: Simple and efficient
- Disadvantage: Not very accurate (first-order approximation)

Let Δt denote a small time step. We would like to compute $s(\Delta t)$ as

$$s(\Delta t) = s(0) + \int_{h=0}^{h=\Delta t} f(s(h), u) dh$$

Fourth-order Runge-Kutta integration:

$$s(\Delta t) \approx s(0) + \frac{\Delta t}{6}(w_1 + w_2 + w_3 + w_4)$$

where

$$w_1 = f(s(0), u)$$

$$w_2 = f(s(0) + \frac{\Delta t}{2} w_1, u)$$

$$w_3 = f(s(0) + \frac{\Delta t}{2} w_2, u)$$

$$w_4 = f(s(0) + \Delta t\, w_3, u)$$

Motion-Planning Problem for Systems with Kinodynamics

Given

- State space S
- Control space U
- Equations of motions as differential equations $f : S \times U \rightarrow \dot{S}$
- State-validity function $\textsc{valid} : S \rightarrow \{\texttt{true}, \texttt{false}\}$ for collisions
- Goal function $\textsc{goal} : S \rightarrow \{\texttt{true}, \texttt{false}\}$
- Initial state s_0

Compute a control trajectory $u : [0, T] \rightarrow U$ so resulting state trajectory $s : [0, T] \rightarrow S$ obtained by integration is valid and reaches the goal, i.e.,

$$s(t) = s_0 + \int_{h=0}^{h=t} f(s(t), u(t))dh \qquad (1)$$

$$\forall t \in [0, T] : \textsc{valid}(s(t)) = \texttt{true} \qquad (2)$$

$$\exists t \in [0, T] : \textsc{goal}(s(t)) = \texttt{true} \qquad (3)$$

Decoupled approach

1. Compute a geometric solution path ignoring differential constraints
2. Transform the geometric path into a trajectory that satisfies the differential constraints

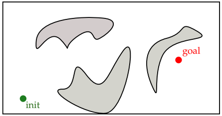
Sampling-based Motion Planning

- Take the differential constraints into account during motion planning

Roadmap Approaches

0. Initialization

add s_{init} and s_{goal} to roadmap vertex set V



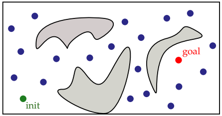
1. Sampling

repeat several times

$s \leftarrow$ StateSample()

if IsStateValid(s) = true

add s to roadmap vertex set V



Roadmap Approaches

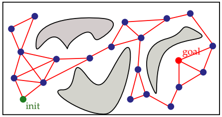
2. Connect Samples

for each pair of neighboring samples
$(s_a, s_b) \in V \times V$

$\lambda \leftarrow \textsc{GenerateLocalTrajectory}(s_a, s_b)$

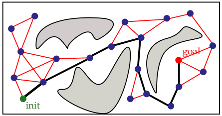
if $\textsc{IsTrajectoryValid}(\lambda) = \texttt{true}$

add (s_a, s_b) to roadmap edge set E



3. Graph Search

search graph (V, E) for path from s_{init} to s_{goal}



$s \leftarrow \textsc{StateSample}()$

- generate random values for all the state components

$\textsc{IsStateValid}(s)$

- place the robot in the configuration specified by the position and orientation components of the state
- check if the robot collides with the obstacles
- check if velocity and other state components are within bounds

$\textsc{IsTrajectoryValid}(\lambda)$

- use subdivision or incremental approach to check intermediate states

Implementation of Roadmap Approaches

$\lambda \leftarrow \textsc{GenerateLocalTrajectory}(s_a, s_b)$

- linear interpolation between s_a and s_b won't work as it does not respect underlying differential constraints
- need to find control function $u : [0, T] \rightarrow U$ such that trajectory obtained by applying u to s_a for T time units ends at s_b
- known as two-point boundary value problem: cannot always be solved analytically, and numerical solutions increase computational cost

Tree Approaches with Differential Constraints

RRT

$\mathcal{T} \leftarrow$ create tree rooted at s_{init}
while solution not found **do**
▷ *select state from tree*
    $s_{\mathrm{rand}} \leftarrow \textsc{StateSample}()$
    $s_{\mathrm{near}} \leftarrow$ nearest configuration in $\mathcal{T}$ to q_{rand} according to distance ρ
▷ *add new branch to tree from selected configuration*
    $\lambda \leftarrow \textsc{GenerateLocalTrajectory}(s_{\mathrm{near}}, s_{\mathrm{rand}})$
    if $\textsc{IsSubTrajectoryValid}(\lambda, 0, \texttt{step})$ **then**
        $s_{\mathrm{new}} \leftarrow \lambda(\texttt{step})$
        add configuration s_{new} and edge $(s_{\mathrm{near}}, s_{\mathrm{new}})$ to $\mathcal{T}$
▷ *check if a solution is found*
        if $\rho(s_{\mathrm{new}}, s_{\mathrm{goal}}) \approx 0$ **then**
            return solution trajectory from root to s_{new}

✓ StateSample(): random values for state components
✓ $\rho : S \times S \rightarrow \mathbb{R}^{\geq 0}$: distance metric between states
✓ IsSubTrajectoryValid($\lambda, 0,$ step): incremental approach

$\lambda \leftarrow$ GenerateLocalTrajectory($s_{\mathrm{near}}, s_{\mathrm{rand}}$)

- will it not create the same two-boundary value problems as in PRM?
- is it necessary to connect to s_{rand}?
- would it suffice to just come close to s_{rand}?

Avoiding Two-Boundary Value Problem

Rather than computing a trajectory from s_{near} to s_{rand} compute a trajectory that starts at s_{near} and extends toward s_{rand}

Approach 1 – extend according to random control

- Sample random control u in U
- Integrate equations of motions when applying u to s_{near} for Δt units of time, i.e.,

$$\lambda \to s(t) = s_{\mathrm{near}} + \int_{h=0}^{h=\Delta t} f(s(t), u)dh$$

Approach 2 – find the best-out-of-many random controls

1. for $i = 1, \ldots, m$ do
   1. $u_i \leftarrow$ sample random control in U
   2. $\lambda_i \to s(t) = s_{\mathrm{near}} + \int_{h=0}^{h=\Delta t} f(s(t), u_i)dh$
   3. $d_i \leftarrow \rho(s_{\mathrm{rand}}, \lambda_i(\Delta t))$
2. return λ_i with minimum d_i

[movie: Traj]

Tree approaches require only the ability to simulate robot motions

- Physics engines can be used to simulate robot motions
- Physics engines provide greater simulation accuracy
- Physics engines can take into account friction, gravity, and interactions of the robot with objects in the evironment

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[movie: PhysicsTricycle]

[movie: PhysicsBug]