Lecture: Analysis of Algorithms (CS583 - 004)

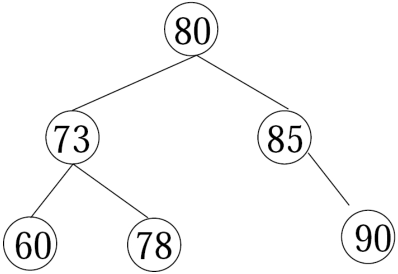
Amarda Shehu

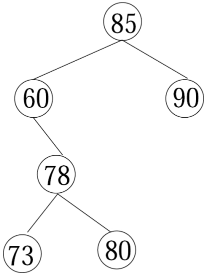
Spring 2019

Definition of a Binary Search Tree

A Binary Search Tree (BST) is a binary tree where:

- if y is a node in the left subtree of x, then key[y] $\leq$ key[x]
- if y is a node in the right subtree of x, then key[y] $\geq$ key[x]





Applications of BSTs

- BSTs are a family of data structures
- Used in online settings, where one needs to maintain and support binary search and other operations on a dynamic set of (ordered) keys
- Binary Space Partition Tree (BSP) – used in almost all 3D video games to determine which objects need to be rendered (are visible from front to back with respect to a viewer at a given location)
- Hash/Merkel tree/trie – used to implement sets and maps, replacing hash tables in functional programming, p2p programs, and specialized image-signatures in which a hash needs to be verified, but whole file not available
- Goldreich-Godlwasser-Micali (GGM) Trees – used in cryptographic applications and complexity theory to generate pseudo-random numbers
- Huffman Coding Tree – used in compression algorithms in .jpeg and .mp3 file formats (will see it in detail in our Greedy Paradigm lecture)
- (Abstract syntax)/Syntax tree – used to represent the abstract syntactic structure of source code written in a programming language
- T-tree – used by main-memory databases, such as Datablitz, EXtremeDB, MySQL Cluster, Oracle TimesTen and MobileLite

Operations to Support on BSTs

- Traversals/Walks
- Querying
- Insertion
- Deletion

BST Traversals or Walks

BST Walks

- **Preorder:** Visit node, then left subtree, then right subtree.
- **Inorder:** Visit left subtree, then node, then right subtree.
- **Postorder:** Visit left subtree, then right subtree, then node.

Inorder Tree Walk

```
Inorder-Tree-Walk(x)
1: if $x \neq NULL$ then
2:     Inorder-Tree-Walk(left[$x$])
3:     print key[$x$]
4:     InorderTree-Walk(right[$x$])
```

- What changes in the other algorithms?
- What is the running time?
- What is the recurrence relationship?
- How would you solve the recurrence?

Querying a BST: Searching for a Key

Querying using Recursion:

```
Tree-Search(x, k)
1: if $x$ = NULL or $k$ =key[$x$] then
2:     return $x$
3: if $k$ <key[$x$] then
4:     return Tree-Search(left[$x$], $k$)
5: else return
   Tree-Search(right[$x$], $k$)
```

Querying using No Recursion:

```
Iterative-Tree-Search(x, k)
1: while $x \neq$ NULL and $k \neq$key[$x$]
   do
2:     if $k$ <key[$x$] then
3:         $x \leftarrow$left[$x$]
4:     else $x \leftarrow$ right[$x$]
5: return x
```

- What do the algorithms return when k is not in the BST?
- What is the running time of these algorithms?
- What is the recurrence relationship of the recursive version?
- How can you solve the recurrence?
- Which implementation would you choose?

Querying a BST: More Operations

Searching for Minimum and Maximum Keys

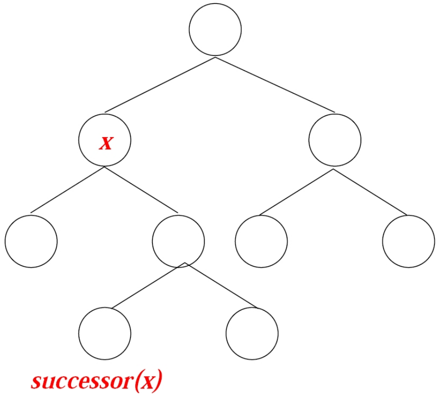
- How would you search for the minimum key in a BST?
- Similarly, how would you search for the maximum key?
- What time cost do you incur on each?

Searching for Successors and Predecessors

- Define successor(x) as the node with smallest key $\geq$ key[x]
- Define predecessor(x) as the node with largest key $\leq$ key[x]
- Design an algorithm to find the successor of a node x
- Design an algorithm to find the predecessor of a node x
- What time cost do you incur on each?

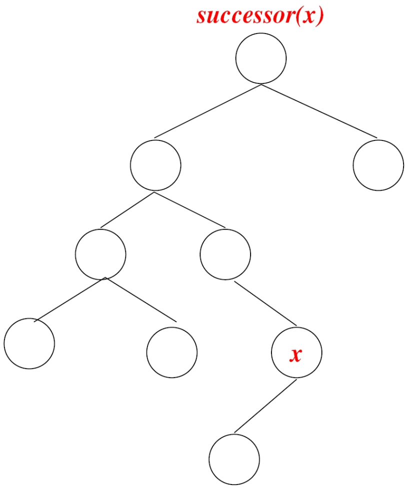
Finding the Successor of a Node: Case 1

- Successor when right subtree not empty:
  - is leftmost node of right subtree
  - has minimum key in right subtree



Finding the Successor of a Node: Case 1

- Successor when right subtree is empty:
  - lowest ancestor of x
  - whose left child is also an ancestor of x



Finding the Successor of a Node

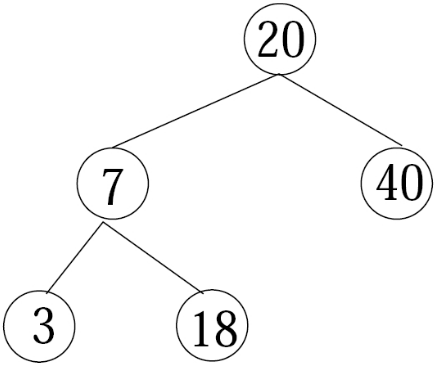
Successor(x)

```
1: if right[$x$] $\neq$ NULL then
2:     return Tree-Minimum(right[$x$])
3: $y \leftarrow p[x]$
4: while $y \neq$ NULL and $x =$right[$y$] do
5:     $x \leftarrow y$
6:     $y \leftarrow p[y]$
7: return $y$
```

- What is the running time of this algorithm?
- How is finding the predecessor different?

Inserting in a BST

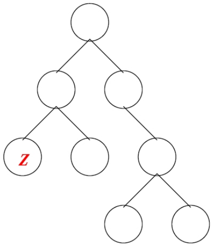
- **Question:** How would you insert a key in a BST?
- **Answer:** In its proper place.

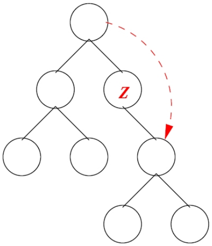


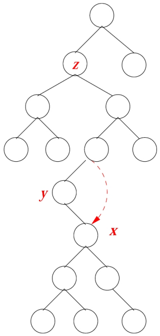
- Insert 10, 21, 24, 30 into BST
- What is the worst-case running time?

Deleting a Node From a BST

- There are three possible cases when removing a node z:
  1. z has no children: delete it
  2. z has only one child: splice it out
  3. z has two children:
     - find the successor y of z
     - splice out y
     - replace the key of z with that of y

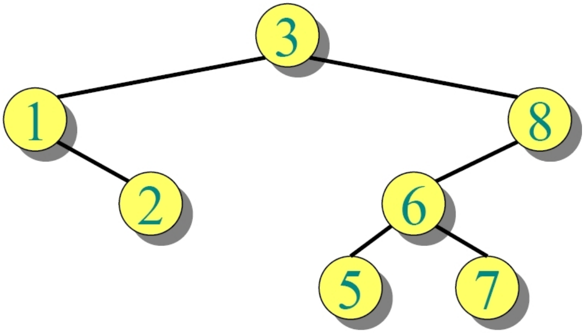






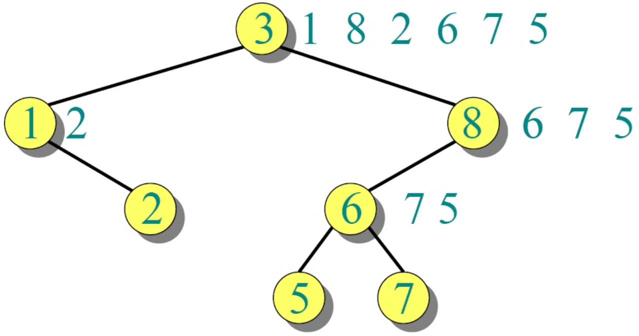
Sorting an Array with BST

- Consider the array $A = [3, 1, 8, 2, 6, 7, 5]$
- Sorting it with a BST requires only two stages:
  1. First insert all the elements in a BST
  2. Then perform an inorder traversal



Time Complexity of Sorting an Array with BST

- BST sort performs the same comparisons as quicksort, but in a different order
- So, the expected time to build the tree is asymptotically the same as the running time of quicksort



Average Node Depth in a BST

- The depth of a node is the number of comparisons made during its insertion into the BST
- A total of $O(n \cdot lgn)$ comparisons are made (expected running time of quicksort)
- So, average depth of a node is $\frac{1}{n} \cdot O(n \cdot lgn)$, or $O(lgn)$

Definition of a Balanced Search Tree

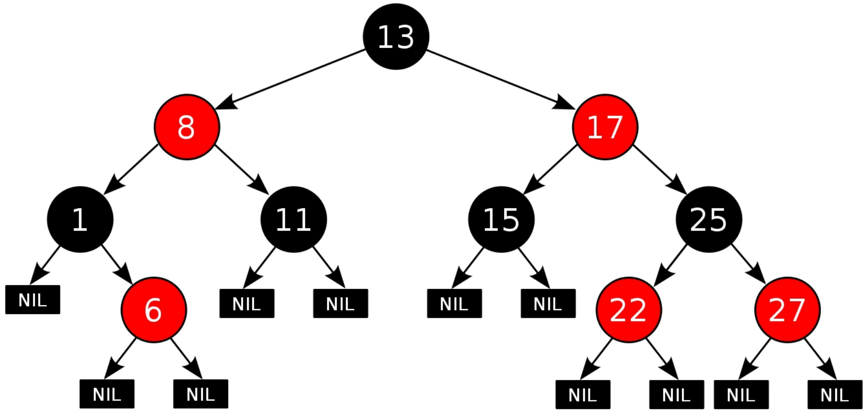
Balanced Search Tree:

A search-tree data structure for which a height of $O(lgn)$ is guaranteed when inserting a dynamic set of n items.

Examples

- AA Trees
- Splay Trees
- Scapegoat Trees
- Treaps
- T-tree
- AVL Trees
- 2-3 and 2-3-4 trees
- B-trees
- **Red-black trees**

Example of a Red-black tree



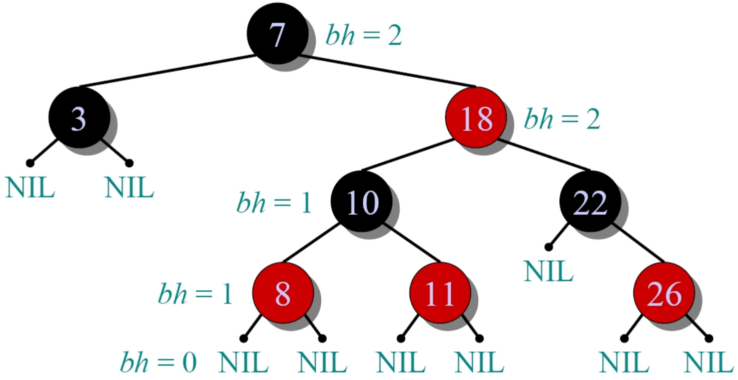
Red-black Trees

Red-black trees are self-balancing binary search trees: they keep their height *small* through transformations known as rotations.

Properties

- Each node is either **red** or **black**
  - A one-bit color field is needed per node
- The root is **black**
- The leaves are the NIL's and are colored **black**
- If a node is **red**, then both its children are **black**
  - This also means that a **red** node has a **black** parent
- All paths from a node to descendant leaves have the same number of **black** nodes (black-height)
  - Property ensures that the tree is balanced

Properties of Red-black Trees



- For convenience, NILs sometimes collected in a NIL sentinel
- Nr. of black nodes on any path from a node x (not including x) down to a leaf is black-height of x, or $\mathrm{bh}(\mathrm{x})$

Height of a Red-black Tree

Theorem: A red-black tree with n internal nodes has height

$$h \leq 2 \cdot lg(n+1)$$

Proof: Makes use of two corollaries

1. The height h of a tree is at most twice the black-height bh
2. Subtree rooted at a node x has $\geq 2^{\mathrm{bh(x)}} - 1$ internal nodes

1. Since a red parent can only have black children, at least half the nodes on a path from root to leaf are black: $\mathrm{bh} \geq \mathrm{h}/2$.

Height of a Red-black Tree (continued)

2. A subtree rooted at x has n_x internal nodes, which are x and the number of internal nodes in its left and right subtrees. Let them be n_l and n_r, respectively. Hence, $n_x = 1 + n_l + n_r$.

The left and right subtrees each have $\geq \mathrm{bh(x)} - 1$ black heights (property of red-black tree). Assuming that the property holds for the left and right subtrees (we are using induction), then $n_x \geq 1 + 2 * (2^{\mathrm{bh(x)}-1} - 1) = 2^{\mathrm{bh(x)}} - 1$. [what is the base case?]

Putting it all together: When x is root, $n_x = n$. So, $n \geq 2^{\mathrm{bh(x)}} - 1$. This means that $lg(n+1) \geq \mathrm{bh} \geq \mathrm{h}/2$. Hence, $h \leq 2 * lg(n+1)$.

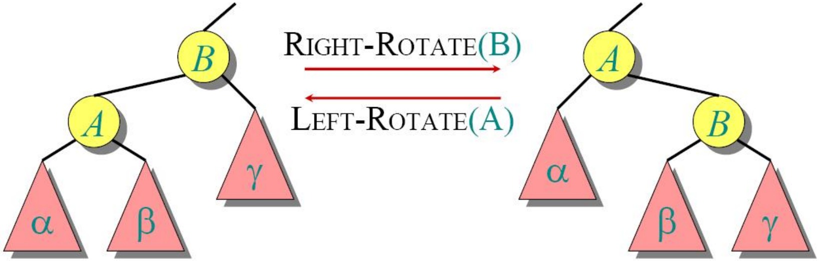
Operations on Red-black Trees

Query Operations: SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR run in $O(lgn)$ time on a red-black tree with n internal nodes

Modifying Operations: INSERT and DELETE, also running in $O(lgn)$ time on the number of internal nodes, cause modifications to the red-black tree

- color changes
- red-black property may be violated
    - restored by restructuring the links of the tree via *rotations*
    - rotations restore the $O(lgn)$ bound of the height

Rotations in a Red-black Tree



Rotations maintain the inorder ordering of keys (BST property):

- $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \leq A \leq b \leq B \leq c$

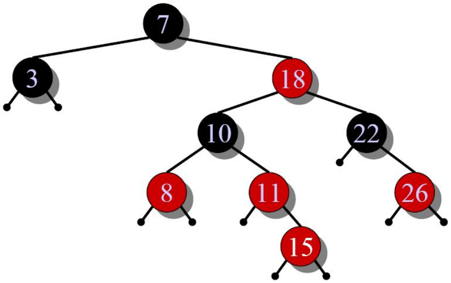
A rotation can be performed in $O(1)$ time.

Insertion into a Red-black Tree

Idea: Insert x in tree and color it red. Only red-black property may be violated. Move the violation up the tree recoloring it until can be fixed with rotations and recoloring.

Example:

- Insert $x = 15$ as in BST
- Recolor and move violation up the tree



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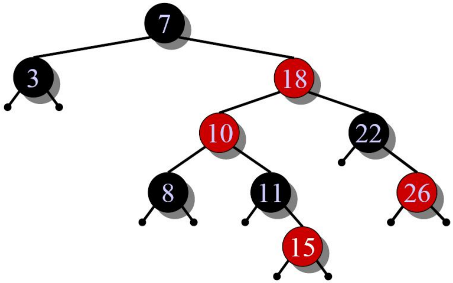

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Example:

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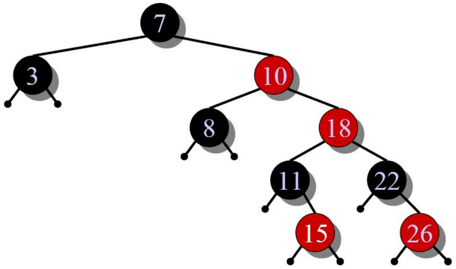
- Right-rotate(18)
- Left-rotate(7) and recolor



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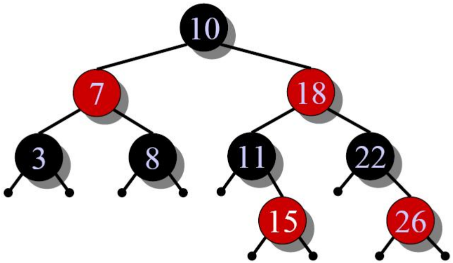


Final Words on Red-black Trees

Insertion running time: $O(lgn)$ with $O(1)$ rotations

Delete has same asymptotic running time and number of rotations

Conclusion: Red-black trees are very useful self-balancing binary search trees to implement associative arrays (e.g., STL map uses such trees). Implementing Insert and Delete for them is complex (and almost always requires looking up material) but pays off.

Binomial Trees

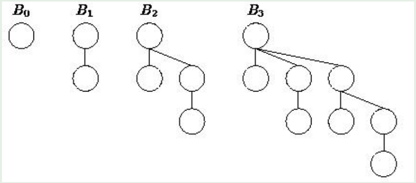
Definition of a Binomial Tree B_k

The binomial tree of order/degree k is defined as follows:

- The binomial tree of order $k = 0$ consists of a single node, r.
- The binomial tree of order $k > 0$ consists of root r and k binomial subtrees $\{B_0, \ldots, B_{k-1}\}$.

Applications and Examples of Binomial Trees

- Price options, dividends, interest rates
- Construct binomial heaps



Properties of Binomial Trees

1. $\text{degree}(\text{root}(B_k)) = k$
2. $|B_k| = 2^k$
3. $h(B_k) = k$

These properties mean that the degree and depth of a binomial tree with n nodes is $lg(n)$.

Number of Nodes in a Binomial Tree

Theorem: The binomial tree B_k has 2^k nodes

Proof: Let $n_k = |B_k|$. We proceed by induction.
Base Case: $|B_0| = 1 = 2^0$ So, the formula holds.
Inductive Step: Assume that the formula holds for all binomial trees of order 0 to $k-1$. By definition, $B_k = \{r, B_0, B_1, \ldots, B_{k-1}\}$. Hence, the number of nodes in B_k is:

$$\begin{aligned} n_k &= 1 + \textstyle\sum_{i=0}^{k-1} B_i \\ &= 1 + \textstyle\sum_{i=0}^{k-1} 2^i \\ &= 1 + \tfrac{2^k-1}{2-1} \\ &= 2^k \end{aligned}$$

We have shown by induction that $n_k = 2^k, \forall k \geq 0$. It follows that binomial trees only come in sizes that are powers of 2. Moreover, for a given power of 2, there is a unique binomial tree.

Height of a Binomial Tree

Theorem: The height of B_k is k.

Proof: Let h_k be the height of B_k. We proceed by induction.
Base Case: By definition, B_0 consists of a single node. So, its height is 0, same as its degree. So, the formula holds.
Inductive Step: Assume that the formula holds for all binomial trees of order 0 to $k-1$. By definition,
$B_k = \{r, B_0, B_1, \ldots, B_{k-1}\}$. Hence, the height of B_k is:

$$\begin{aligned} h_k &= 1 + \max_{0 \leq i \leq k-1} h_i \\ &= 1 + \max_{0 \leq i \leq k-1} i \\ &= 1 + k - 1 \\ &= k \end{aligned}$$

Hence, we have shown by induction that $h_k = k$, $\forall k \geq 0$. Since $|B_k| = 2^k$ and $h_k = k$, then $h_k = lg(|B_k|)$.

Alternative Ways of Constructing Binomial Trees

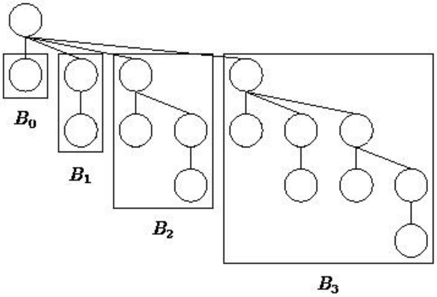


Figure: $B_k = \{r, B_0, \ldots, B_{k-1}\}$

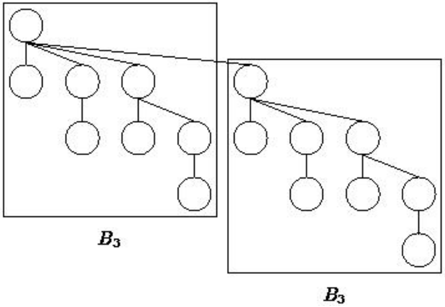


Figure: $B_k = \{B_{k-1}, B_{k-1}\}$

Connection between Binomial Tree and Binomial Theorem

What is the origin of the name *binomial tree*?

- Number of nodes at a given depth in a binomial tree is determined by the binomial coefficient.
- The binomial coefficient is in the context of the binomial theorem.
- The binomial theorem computes the power of a binomial.
- A binomial is an expression that consists of two terms, x+y. That is why $B_k = \{B_{k-1}, B_{k-1}\}$ is called a binomial tree.

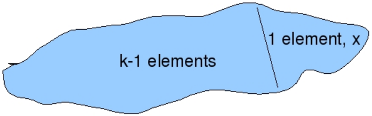
Binomial Theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^i \cdot y^{n-i}$, where $\binom{n}{i}$ is the binomial coefficient.

Properties of Binomial Trees

Theorem: The number of nodes at depth l in B_k, where $0 \leq l \leq k$ is given by the binomial coefficient $\binom{k}{l}$

Proof: Let $n_k(l)$ be the number of nodes at depth l in B_k. We now proceed by induction.
Base Case: Since $n_k(0) = 1 = \binom{k}{0} = 1$, the formula holds.
Inductive Step: Assume that $n_i(l) = \binom{i}{l}$ for $0 \leq i \leq k-1$. Since $B_k = \{B_{k-1}, B_{k-1}\}$, we have:

$$\begin{aligned} n_k(l) &= n_{k-1}(l) + n_{k-1}(l-1) \\ &= \binom{k-1}{l} + \binom{k-1}{l-1} \\ &= \frac{(k-1)!(k-1+1-l)+(k-1)!l!}{(k-1+1-l)!l!} \\ &= \binom{k}{l} \end{aligned}$$

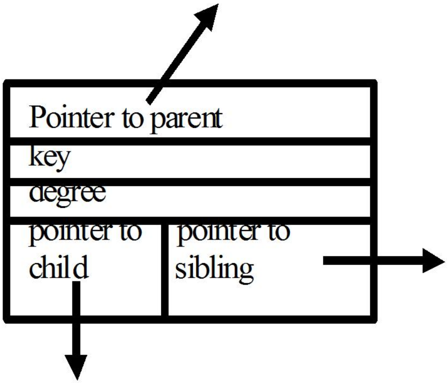


Two options for choosing l out of k: include x (so choose l-1 out of k-1) or not (so choose all l out of k-1)

We showed by induction that $n_k(l) = \binom{k}{l}$

Basic Node Implementation in Binomial Trees

- Maintaining pointer to parent allows up-traversal
- Maintaining degree to know how many children
- Maintaining pointer to sibling allows in-traversal
- Implementation affects Merge operation

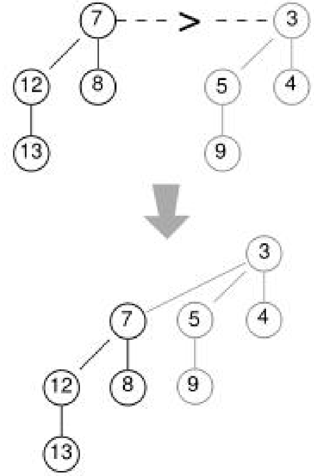


Merging Two Binomial Trees of Same Order

- Consider binomial trees that satisfy the min-heap property
- Merging two binomial trees of same order is an important operation: $B_{k+1} \leftarrow B_k + B_k$
- Its running time is $O(1)$

MERGETrees(p, q)

```
1: if key[p.root] ≤ key[q.root] then
2:     return p.addSubTree(q)
3: else return q.addSubTree(p)
```



Binomial Heaps

A binomial heap is a set of binomial trees with the two properties:

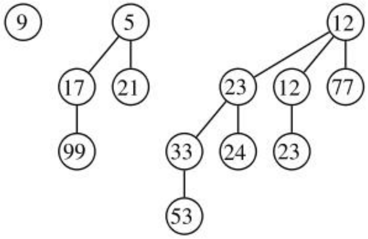
1. Each binomial tree in a heap obeys the minimum-heap property: the key of parent is $\leq$ than the keys of the children.
    - This property ensures that the root of each binomial tree contains the smallest key in the tree.
2. There are ≤ 1 binomial trees for each order.
    - This property implies that a binomial heap with n elements has $\leq lgn + 1$ binomial trees.

Number and Orders of Binomial Trees in a Binomial Heap

- The number and orders of these trees are uniquely determined by the number of elements n.
- Each binomial tree corresponds to digit one in the binary representation of number n.

E.g., 13 is 1101 in binary, which is $2^3 + 2^2 + 2^0$.

A binomial heap with 13 elements consists of three binomial trees of orders 3, 2, and 0.

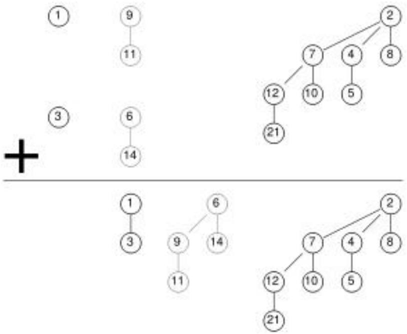


Operations on Binomial Heaps

- Insert new element: $O(lgn)$
    - Create a new heap containing element: $O(1)$
    - Merge new heap with current one: $O(lgn)$
- Find minimum: $O(lgn)$
    - Find minimum among roots of binomial trees
    - There are $O(lgn)$ binomial trees
- Delete minimum: $O(lgn)$
    - Find minimum element: $O(lgn)$
    - Remove element from its binomial tree: $O(1)$
    - Obtain list of subtrees: $O(1)$
    - Reorder from largest to smallest: $O(lgn)$
    - Merge heap with original heap: $O(lgn)$
- Merge two heaps: $O(lgn)$

Merging Two Binomial Heaps

- Lists of roots of heaps are traversed simultaneously
- If only one heap contains B_j, move B_j to merged heap
- If both heaps H_1 and H_2 contain B_j, create $B_{j+1} \leftarrow B_j + B_j$
- Alternatively, all roots can be linked in a linked list in sorted order
- Iterate over list and merge binomial trees of same order



- Heaps have $\leq \lfloor lg(n_1) \rfloor + 1$ and $\lfloor lg(n_2) \rfloor + 1$ roots
- $T(n)$ is then $O(lgn)$ calls to MergeTrees $\rightarrow O(lgn)$.