Lecture 8: (Predicate) First Order Logic

CS 580 (001) - Spring 2018

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Pros and Cons of Propositional Logic

- 😈 Propositional logic is **declarative**: pieces of syntax correspond to facts
- 😈 Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- 😈 Propositional logic is **compositional**: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- 😈 Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power (unlike natural language) E.g., cannot say "pits cause breezes in adjacent squares" except by writing one sentence for each square

First-order Logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries ...
- Relations: red, round, bogus, prime, multistoried ...,
brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
- Functions: father of, best friend, third inning of, one more than, end of ...

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic Elements

Constants	$KingJohn,\ 2,\ GMU, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Longrightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic Sentences

Atomic sentence $=$ $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $=$ $function(term_1, \ldots, term_n)$
or *constant* or *variable*

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex Sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \implies S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \implies Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg >(1,2)$

Truth in First-order Logic

Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

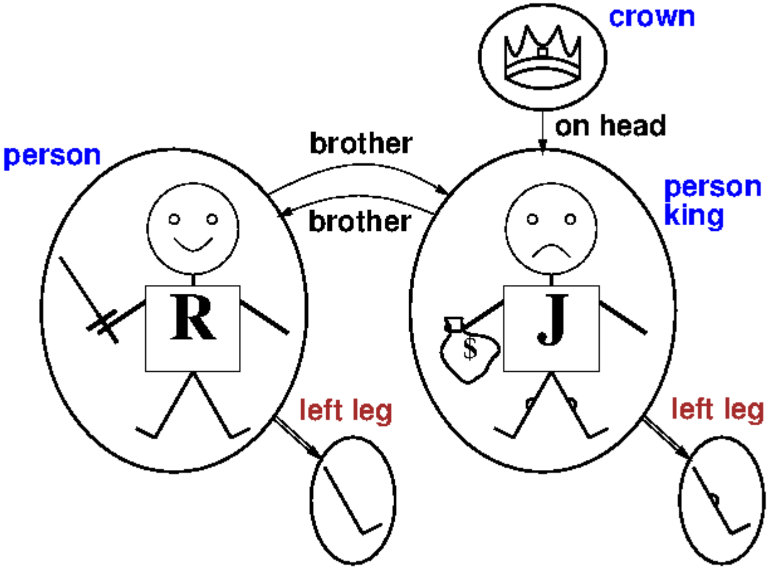
Interpretation specifies referents for
constant symbols $\rightarrow$ objects

predicate symbols $\rightarrow$ relations

function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by *predicate*

Models for FOL: Example



Truth Example

Consider the interpretation in which:
Richard $\rightarrow$ Richard the Lionheart
John $\rightarrow$ the evil King John
Brother $\rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
  For each k-ary predicate P_k in the vocabulary
    For each possible k-ary relation on n objects
      For each constant symbol C in the vocabulary
        For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal Quantification

$\forall \; \langle variables \rangle \langle sentence \rangle$

Everyone at GMu is smart:
$\forall \; x\, At(x, GMU) \implies Smart(x)$
$\forall \; x\, P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, GMU) \implies Smart(KingJohn)) \\
\wedge \quad & (At(Richard, GMU) \implies Smart(Richard)) \\
\wedge \quad & (At(GMU, GMU) \implies Smart(GMU)) \\
\wedge \quad & \ldots
\end{aligned}
$$

A common Mistake to Avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$\forall x \quad At(x,GMU) \wedge Smart(x)$ means "Everyone is at GMU and everyone is smart"

$\exists \ \langle variables \rangle \langle sentence \rangle$

Someone at Stanford is smart:
$\exists \ x \, At(x, Stanford) \land Smart(x)$
$\exists \ x \, P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{array}{ll}
 & (At(KingJohn, Stanford) \land Smart(KingJohn)) \\
\lor & (At(Richard, Stanford) \land Smart(Richard)) \\
\lor & (At(Stanford, Stanford) \land Smart(Stanford)) \\
\lor & \ldots
\end{array}
$$

Another Common Mistake to Avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\implies$ as the main connective with $\exists$:

$\exists x \quad At(x,Stanford) \implies Smart(x)$
is true if there is anyone who is not at Stanford!

Properties of Quantifiers

$\forall \ x \ \ \forall \ y$ is the same as $\forall \ y \ \ \forall \ x$

$\forall\ x\ \ \forall\ y$ is the same as $\forall\ y\ \ \forall\ x$ (why??)

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"There is a person who loves everyone in the world"

$\forall\ y\ \ \exists\ x\ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other
$\forall\ x\ Likes(x, IceCream) \qquad \neg\exists\ x\ \neg Likes(x, IceCream)$
$\exists\ x\ Likes(x, Broccoli) \qquad \neg\forall\ x\ \neg Likes(x, Broccoli)$

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One's mother is one's female parent

$\forall \; x, y \; Mother(x, y) \;\; \Leftrightarrow \;\; (Female(x) \wedge Parent(x, y))$

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$\forall \; x, y \; FirstCousin(x, y) \;\Leftrightarrow\; \exists \; p, ps \; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

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Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall\ x\ \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) *Sibling* in terms of *Parent*:

$$\forall\ x, y\ Sibling(x, y)\ \Leftrightarrow\ [\neg(x = y) \wedge \exists\ m, f \neg(m = f) \wedge$$

$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists \; a \; Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

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Answer: Yes, $\{a/Shoot\}$ ← substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$
$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

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"Perception"
$\forall\ b, g, t\ Percept([Smell, b, g], t) \implies Smelt(t)$
$\forall\ s, b, t\ Percept([s, b, Glitter], t) \implies AtGold(t)$

Reflex: $\forall\ t\ AtGold(t) \implies Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall\ t\ AtGold(t) \wedge \neg Holding(Gold, t) \implies Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

Properties of locations:
$\forall \; x, t \; At(Agent, x, t) \wedge Smelt(t) \implies Smelly(x)$
$\forall \; x, t \; At(Agent, x, t) \wedge Breeze(t) \implies Breezy(x)$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect
$\forall \; y \; Breezy(y) \implies \exists \; xPit(x) \wedge Adjacent(x, y)$

Causal rule—infer effect from cause
$\forall \; x, y \; Pit(x) \wedge Adjacent(x, y) \implies Breezy(y)$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

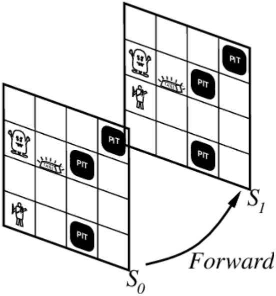
Definition for the *Breezy* predicate:
$\forall \; y \; Breezy(y) \; \Leftrightarrow [\exists \; xPit(x) \wedge Adjacent(x, y)]$

Keeping Track of Change

Facts hold in situations, rather than eternally
E.g., *Holding*(*Gold*, *Now*) rather than just *Holding*(*Gold*)

Situation calculus is one way to represent change in FOL:
Adds a situation argument to each non-eternal predicate
E.g., *Now* in *Holding*(*Gold*, *Now*) denotes a situation

Situations are connected by the *Result* function
Result(*a*, *s*) is the situation that results from doing *a* in *s*



Situation calculus is a logic formalism designed for representing and reasoning about dynamical domains.

a dynamic world is modeled as progressing through a series of situations as a result of various actions being performed within the world.

Introduced by John McCarthy in 1963.

McCarthy described a situation as a state.

Ray Reiter corrected it in 1991:

A situation is a finite sequence of actions. Period. It's not a state, it's not a snapshot, it's a history. ["Situation Calculus Ontology"]

Describing Actions I

"Effect" axiom—describe changes due to action
$\forall\ \ s\ AtGold(s) \implies Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe **non-changes** due to action
$\forall\ \ s\ HaveArrow(s) \implies HaveArrow(Result(Grab, s))$

Frame problem: find an elegant way to handle non-change
(a) representation—avoid frame axioms
(b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

Describing Actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

P true afterwards ⇔ [an action made P true
∨ P true already and no action made P false]

For holding the gold:

$$\forall\ a, s\ Holding(Gold, Result(a, s)) \Leftrightarrow [(a = Grab \wedge AtGold(s)) \vee (Holding(Gold, s) \wedge a \neq Release)]$$

Making Plans

Initial condition in KB:

$At(Agent, [1,1], S_0)$
$At(Gold, [1,2], S_0)$

Query: $Ask(KB, \exists\ s\ Holding(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists\ p\ Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:
$\forall\ s PlanResult([\,], s) = s$
$\forall\ a, p, s\ PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

First-order logic:
- objects and relations are semantic primitives

- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB