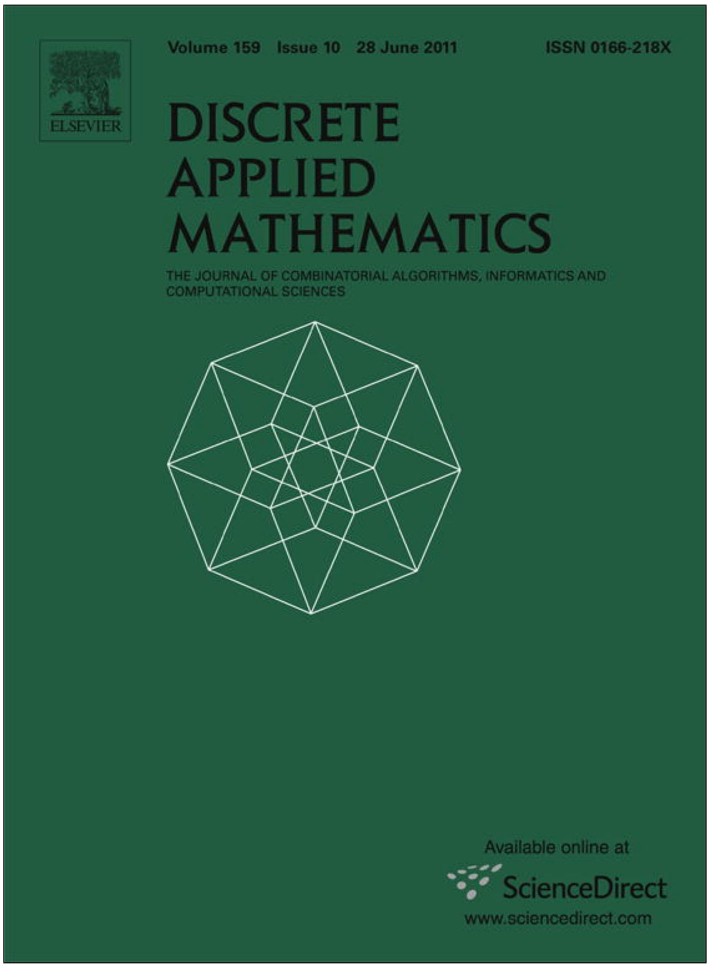
Volume 159 Issue 10 28 June 2011
ISSN 0166-218X
ELSEVIER
DISCRETE APPLIED MATHEMATICS
THE JOURNAL OF COMBINATORIAL ALGORITHMS, INFORMATICS AND COMPUTATIONAL SCIENCES
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Applied Mathematics 159 (2011) 1002–1012

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Some special minimum k-geodetically connected graphs

Yingjie Lan [a,*], Songqing Chen [b]

[a] Peking University, China
[b] George Mason University, USA

ARTICLE INFO

Article history:
Received 30 July 2009
Received in revised form 11 February 2011
Accepted 6 March 2011
Available online 31 March 2011

Keywords:
Minimum size graph
Geodetic connectivity
Distance invulnerability
Robust system design
Survivable network

ABSTRACT

A connected graph G is k-geodetically connected (k-GC) if the removal of less than k vertices does not affect the distances (lengths of the shortest paths) between any pair of the remaining vertices. As such graphs have important applications in robust system designs, we are interested in the minimum number of edges required to make a k-GC graph of order n, and characterizing those minimum k-GC graphs. When $3 < k < (n-1)/2$, minimum k-GC graphs are not yet known in general, even the minimum number of edges $m(n, k)$ is not determined. In this paper, we will determine all of the minimum k-GC graphs for an infinite set of special (n, k) pairs that were formerly unknown. To derive our results, we also developed new bounds on $m(n, k)$. Additionally, we show that k-GC graphs with small relative optimality gaps can be easily constructed and expanded with great flexibilities, which gives convenient applications for robust system designs.

1. Introduction

Today computer systems are integrated into our daily lives. With their ever-increasing complexity, the failure or the malfunction of some (software and/or hardware) elements in these systems is unavoidable, which could substantially impact our lifeif the entire system is brought down. Facing such challenges, many systems have taken robustness into the design. A common thread of the robust system design is to increase the redundancy while minimizing the cost so that the failure of the element will not or will trivially impact the function of running systems, which we refer as failure transparency. Roughly, failure transparency could be defined as that the system can still carry out the function after element failures. In a more strict sense, the failure transparency that is aimed at in our study not only includes the continuous functioning of the system, but also includes the unchanged performance of the system function. Such transparency can only be kept up to a certain amount of element failures: if failed elements accumulate to a certain level, the system performance would start to deteriorate until its eventual final breakdown. Such a requirement has many applications in computer communication systems.

- *Network traffic routing* Routing is one of the most important issues in building networks. In particular, for the Internet, which connects different domains through different ISPs, robust routing is very essential to the availability and reliability of various Internet services that we rely on daily. As the Internet traffic routing is through various gateway routers operated by different Internet service providers, while the malfunction or failure of routers is not uncommon, a significant amount of effort has been devoted to dealing with router deviations [16]. On the other hand, if the Internet routes are constructed or enhanced with strict failure transparency, the severity and the cost of router failure or malfunction could be minimized, a goal of the recent NSF initiatives to build the next generation Internet.

* Corresponding address: Guanghua School of Management, Beijing University, Beijing 100871, China. Fax: +86 1062759353.
E-mail addresses: ylan@gsm.pku.edu.cn (Y. Lan), sqchen@cs.gmu.edu (S. Chen).

0166-218X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2011.03.009

- *Multiple Description Coding (MDC)* As a communication link may be unreliable, MDC has been proposed to increase the robustness of our communication systems. With the rapid increase of Internet media content that commonly demands larger and continuous bandwidth support for media data transmission (than the traditional text-based Web content), MDC has been proposed to code a media stream into two or more complementary descriptions. If each description is received, it could be decoded with low video quality. If all descriptions are received, the original video quality could be achieved. To efficiently realize MDC for Internet media, a natural requirement is to construct multiple (non-overlapping and shortest) paths so that these multiple descriptions can take different routes to the destination. This way the loss or the delay of any description due to congestion or failure will not disrupt the service [1].
- *Failure-resilient wireless sensor networks* Wireless sensor networks have attracted considerable attention from both the research and industrial communities, as they could be used in various applications, such as military surveillance, nuclear factory radiation/wild life/forest fire monitoring. These deployed sensors commonly form an ad-hoc network to communicate the collected information. However, as each sensor is fragile, it is fundamental to have multiple paths existing among these sensors for their communications with each other and to the data collecting center. Considering that these sensors are normally battery powered, the construction of these multiple paths must consider the power consumption as well, which naturally requires the construction of multiple shortest paths. Thus, robust and energy efficient path construction is critical to the success of these applications employing wireless sensor networks [11].

These applications indicate that many systems require robust communications that can survive element failures, which can be naturally modeled using graphs. These have led to the study of various models for survivable networks, among which are k-connected or k-edge-connected graphs. The k-(edge)-connected graphs remain connected within any $k-1$ vertices (edges) failures. However, element failures can increase the distances (the lengths of the shortest paths) between certain pairs of vertices. In many applications, distances are related to performances, therefore it is desirable to have them remain unchanged within any $k-1$ element failures. Those graphs are referred to as the k-geodetically connected (k-GC) graphs or k-geodetically edge-connected (k-GEC) graphs, depending on if the failure of vertices or that of edges are considered. Entringer et al. [8] were the first who studied those graphs.

Notations used here are similar to those in [14]. Given a graph G, $V(G)$ is the set of vertices and $E(G)$ is the set of edges; the order of G (the number of vertices in G) is $n=|V(G)|$, and the size of G is $m=|E(G)|$, the number of edges. The size of a k-GC graph or a given order n often needs to be minimized to reduce the cost for building such a system. A minimum k-GC graph of order n is a k-GC graph whose size is minimized, and the minimum size is denoted as $m(n,k)$. A shortest path between two vertices u and v is called a $u-v$ geodesic, and a set of $u-v$ geodesics is said to be internally disjoint if they share no common vertices except for u and v. The distance between u and v is simply the length of any $u-v$ geodesic, denoted as $d(u,v)$. Given a vertex u of a connected graph G, we denote $D_i^u=\{v\in V(G): d(u,v)=i\}$, the set of vertices with distance i from u, and specifically, $D_0^u=\{u\}$. These sets form a distance decomposition of all the vertices, which are mutually disjoint, and shall also be partite if G is a connected graph. Let r be the maximum index such that $D_i^u\neq\emptyset, \forall i\leq r$, and r is called the eccentricity of u [7,12]. The maximum eccentricity is called the graph diameter [12]. For any vertex u, the neighborhood of u is denoted as $N(u)=\{v\in V(G):(u,v)\in E(G)\}$. If there are no self-loops in the graph, then $u\notin N(u)$. Clearly, if $v\in D_i^u$, then $N(v)\subset D_{i-1}^u\cup D_i^u\cup D_{i+1}^u$. The degree of u is given by $\deg(u)=|N(u)|$, and δ is the minimum of the degrees of all vertices in the graph. If all degrees are the same and equal to s, the graph is said to be s-regular. The notion of neighborhood can be extended to a set of vertices: $S\subset V(G)$, $N(S)=\bigcup_{u\in S}N(u)$. A complete bipartite graph is denoted by $K(s,t)$, whose vertices are from two exclusive subsets V_1 and V_2 with $|V_1|=s$ and $|V_2|=t$, and $V(K)=\{(u,v): u\in V_1, v\in V_2\}$. Finally, $W(s,t)$ denotes a wreath with parameter s and t, which will be defined in detail later.

In this paper, we set out to investigate some special k-GC graphs of minimal sizes that were not known in the literature. A k-GC is a connected graph G such that the removal of up to any $k-1$ vertices does not affect the distances between any pair of the remaining vertices. We are interested in minimizing the size of k-GC graphs of order n, and characterizing them. When $3<k<(n-1)/2$, minimum k-GC graphs are not yet known in general, even the minimum size $m(n,k)$ is hard to determine. We will fully characterize all of the minimum k-GC graphs for an infinite set of special (n,k) pairs that were formerly unknown. To derive our results, new lower and upper bounds of $m(n,k)$ are developed. The relative errors of the new bounds are within 1/8 and can be arbitrarily close to 0 in favorable cases. Accordingly, sub-optimal k-GC graphs within the bounds can be easily constructed with a great amount of varieties on different topologies. Furthermore, it is easy to extend such graphs into larger k-GCs. These discoveries provide significant flexibility in robust systems design, particularly for those communication systems that demand the coexistence of multiple shortest paths for reliable services as aforementioned.

The rest of the paper is organized as follows. In Section 2, we review some of the literature on k-GC graphs. In Section 3, an improved lower bound is given, coupled with an upper bound to help us understand how close those bounds are. Section 4 concentrates on some wreaths as minimum k-GC graphs. In Section 5, we continue on to the full description of all possible minimum graphs besides those special wreaths. We make concluding remarks and discuss some future work in Section 6.

2. Background and related work

In this section, we briefly present some of the fundamental results and most related work to our study. Some of the closely related results are discussed in detail for the reader's convenience.

Definition 1 ([8]). A graph G is said to be k-(edge)-geodetically connected (k-GC or k-GEC for short) if the removal of at least k vertices (edges) is required to increase a distance $d(u, v) \geq 2$ or reduce G to a disconnected graph or a single vertex.

This is the traditional definition. Clearly k-GC and k-GEC are special k-connected graphs, because k-GC and k-GEC graphs require not only connectivity of the graph after the removal of $k - 1$ elements (vertices or edges), but also that distances remain unchanged between any remaining vertices. This additional requirement generates different properties for k-GC and k-GEC graphs. Nevertheless, several properties can be transferred from k-connected graphs to k-GC and k-GEC graphs even in a stronger sense. The following reformulation of the basic result by Entringer et al. [8] is taken from [14]:

Theorem 1 ([14]). *The following statements are equivalent for any graph G and integer $k > 0$.*

(1) *G is k-GC.*
(2) *G is connected if exactly one of the following holds:* (a) *G is a complete graph of order at least $k + 1$;* (b) *G is not complete and every two vertices distance 2 apart are connected by at least k internally disjoint geodesics.*
(3) *G is connected if exactly one of the following holds:* (a) *G is a complete graph of order at least $k + 1$;* (b) *G is not complete and for all $u, v \in V(G)$ any set of fewer than k internally disjoint $u - v$ geodesics of length at least 2 can be completed to a superset of k internally disjoint $u - v$ geodesics.*
(4) *G is k-GEC.*

By Theorem 1, items (1) and (4) tell us that k-GC and k-GEC graphs are equivalent. It suffices to study either of them and then apply the results to the other. In this paper, we will concentrate on k-GC graphs. Items (2) and (3) indicate that any vertex $u \in V(G)$ in a k-GC graph G has at least k neighbors, which requires that the minimum degree δ of G is at least k.

In the past, efforts have been made to find minimum k-GC graphs, and naturally this task has been divided by the connectivity parameter k into sub-tasks. Farley and Proskurowski [10] completely solved the case for $k = 2$. They called 2-GC graphs self-repairing graphs [9], which are analogs of classical blocks and are of special interest. They proved that every 2-GC graph with more than four ($n \geq 4$) vertices has at least $2n - 4$ edges and completely determined the minimum graphs: except for the 3-cube, each such graph is a so-called twin graph. Recently, Bosíková [3] has determined the minimum 3-GC graphs. For digraphs, some good results were obtained by Plesník [15], who found the exact minimum size of a k-GC digraph of order n when $n \bmod k = 0$, and gave quite good bounds for all n.

A special branch of research on minimum k-GC graphs was conducted on graphs of diameter 2. With graphs of diameter 2, the results are quite complete. The case $2k \leq n$ was proved by Bollobás and Eldridge [2], as they studied diameter invulnerability. Then, Jackson and Entringer [13] complemented the result by the case of $2k \geq n - 1$, so that for any (n, k) pair, all minimum k-GC graphs of diameter 2 are completely determined.

Theorem 2 ([2,13]). *Given integers $0 < k < n$, let p and q be integers satisfying $n = (p - 1)(n - k) + q$, where $0 < q \leq n - k$. Then every k-GC graph with order n of diameter at most 2 has at least $(n - q)(k + q)/2$ edges and the complete p-partite graph $K(n - k, \ldots, n - k, q)$ is the only extremal graph.*

But those graphs are generally not minimum k-GC graphs if graphs of arbitrary diameters are considered. In this paper we are going to give complete solutions to a special set of minimum k-GC problems, as defined by their (n, k) pairs. We first look at some bounds on $m(n, k)$ developed in the literature, which are helpful to determine minimum k-GC graphs. Theorem 1 indicates that every vertex of a k-GC graph has a degree of at least k, which gives a trivial lower bound on the size $m(n, k)$: $m(n, k) \geq kn/2$. This bound can be obtained by a complete bipartite graph $K(k, k)$ with $2k$ vertices. Thus they must be minimum k-GC graphs. Interestingly, when k is relatively small compared with n, a special lower bound has been derived [14]:

Theorem 3 ([14]). *For any k-GC graph with $k \geq 1$, minimum degree δ, and order n, we have*

$$m \geq kn - k^2 - (\delta - k)(k - 2).$$

To prove Theorem 3, the following lemma is needed:

Lemma 1 ([14]). *Let G be a k-GC graph with $k \geq 1$, δ its minimum degree and u a vertex of degree δ. If the eccentricity of u is r, then*

$$m \geq kn - k^2 - (\delta - k)(k - 1 - |D_r^u|/2). \tag{1}$$

Moreover, if $\delta - k + 1 \geq |D_r^u|$, then

$$m \geq kn - k^2 - (\delta - k)(k - 1 - |D_r^u|/2) + (\delta - k + 1 - |D_r^u|)|D_r^u|/2. \tag{2}$$

Theorem 3 can be readily derived by discussing two cases of $|D_r^u|$: when $|D_r^u| = 1$, $\delta - k + 1 \geq |D_r^u|$ will be satisfied, and inequality (2) would yield the desired result; otherwise $|D_r^u| \geq 2$, inequality (1) would yield the desired result. The proof of Lemma 1 is based on the distance decomposition of vertices around a vertex u of minimum degree δ. If $r = 1$, then G

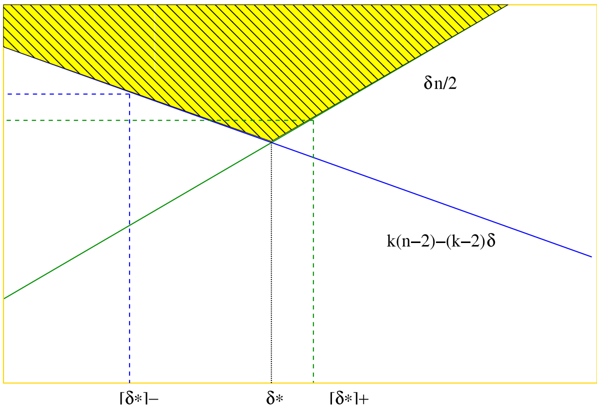

Fig. 1. Minimum obtained where the two terms equal.

must be a complete graph with $\delta = n - 1 \geq k$, which is a trivial case. Thus we assume $r \geq 2$, and there would be exactly δ edges between D_0^u and D_1^u. For each $i = 2, \ldots, r$ and any vertex $v \in D_i^u$, since there are at least k internally disjoint $u - v$ geodesics, there are at least k edges from v to D_{i-1}^u. Hence there are at least $k|D_i^u|$ edges between D_i^u and D_{i-1}^u. For the last set D_r^u, the number of degrees not yet counted is at least $|D_r^u|(\delta - k)$. These degrees may be caused by edges to D_{r-1}^u or by those within D_r^u. Thus each degree contributes at least $1/2$ edge and we have at least $|D_r^u|(\delta - k)/2$ edges. Summing up all those lower estimations, we have inequality (1). If $\delta - k > |D_r^u| - 1$, since at most $|D_r^u| - 1$ degrees can belong to edges within D_r^u, the remaining degrees must come from edges running into D_{r-1}^u, and they shall correspond to a whole edge instead of a half edge as already counted before, which yields an additional term of $(\delta - k + 1 - |D_r^u|)|D_r^u|/2$ in inequality (2).

3. A lower bound without δ

As the lower bound in Theorem 3 depends on an unknown δ, a lower bound free from δ would be more desirable for the search of minimum k-GC graphs. A modest improvement on the lower bound in Theorem 3 can be made: given the minimum degree δ of graph G, one can immediately tell that the number of edges m of G must satisfy that $m \geq n\delta/2$. Consider that together with the lower bound in Theorem 3, we have:

Lemma 2. *For any k-GC graph with $k \geq 1$, minimum degree δ, and order n, we have $m \geq \max(n\delta/2, k(n-2) - \delta(k-2))$.*

Here the second expression in Theorem 3 is simplified and canceled out the term k^2. Lemma 2 provides a lower bound that is no longer monotonic with regard to δ. Initially, when δ starts from k and increases, the lower bound decreases; after it passes a certain point, the lower bound starts to increase as δ increases. Fig. 1 illustrates such a situation. Clearly, there exists an optimal integer value for δ, such that the lower bound reaches the minimum, which would provide a theoretical lower bound for k-GC graphs free from δ! Such a lower bound is clearly the optimal objective of the following integer optimization problem,

$$\min_{\delta} \max(n\delta/2, k(n-2) - \delta(k-2)).$$

Solve the relaxed optimization first, where δ need not be an integer. If $k > 2$, $k(n-2) - \delta(k-2)$ is decreasing in δ, while $n\delta/2$ is increasing in δ. As both terms are linear, it is a convex optimization, and the two terms in the objective must equal each other at optimum, by which an optimal solution δ^* is found:

$$n\delta^*/2 = k(n-2) - \delta^*(k-2) \Longrightarrow \delta^* = \frac{k(n-2)}{n/2 + k - 2}.$$

If δ^* is not an integer, check the $\lfloor \delta^* \rfloor$ and $\lceil \delta^* \rceil$. Compare the objectives at both integer values and the one with smaller objective value is the optimal solution to the integer optimization problem. Those steps will also lead to the right answer if δ^* is an integer, as the optimal solution is simply δ^*. Thus, another lower bound for the size of k-GC graphs is found as stated in the next theorem.

Theorem 4. *For any k-GC graph with $n > k > 2$, we have a lower bound for the size $m(n, k)$:*

$$l(n, k) = \min(\lceil \delta^* \rceil n/2, k(n-2) - \lfloor \delta^* \rfloor (k-2)),$$

where

$$\delta^* = \frac{k(n-2)}{n/2 + k - 2}.$$

As expected, with a δ^* that can be readily computed from n and k, now δ is discarded. And Fig. 1 shows that $\delta^* n/2 \leq l(n, k)$, so $\delta^* \leq 2m(n, k)/n$, i.e. δ^* is a lower bound for the average vertex degree, which in turn is a lower bound for the maximum vertex degree σ. Clearly, $k < \delta^* < 2k$ as $n/2 > k > 1$, direct computation yields $\delta^* > 2k - 1$ if $n > 2(k-1)(2k-1) + 2$. That shows this interesting corollary below is true:

Corollary 1. *For any k-GC graph with $k > 2$, if $n > 2(k-1)(2k-1) + 2$, the maximum vertex degree σ is at least $2k$.*

To see the effectiveness of the bound $l(n, k)$, the following items should be investigated: (1) the relative performance as measured by either the ratio $m(n, k)/l(n, k)$ or the difference $m(n, k) - l(n, k)$; (2) instances that can actually obtain the lower bound. The remaining part of this section addresses the first item, and the second item is left for the next one.

The following theorem gives insights on the internal structure of a k-GC graph. The theorem is also known as the neighborhood test, which is essentially item (2) of Theorem 1, and the proof can be found in [8].

Theorem 5 (*[8]*)**.** *A connected graph G with order $n > k$ is a k-GC graph if and only if any two vertices that are not directly connected by an edge in G have either no common neighbors or at least k common neighbors.*

Theorem 5 provides a simple way to compose new k-GC graphs: if G is a k-GC graph of order n, and $u \in V(G)$, construct G' as

$$V(G') = V(G) \cup \{u'\};$$
$$E(G') = E(G) \cup \{(v, u') : (v, u) \in E(G)\}.$$

If G can pass the neighborhood test, so can G', therefore it is also a k-GC graph. This operation, called vertex cloning, was suggested in [9]. Another method to add a vertex to an existing k-GC is called isosceles extension, which was suggested by Chang et al. [6]. In isosceles extension, they gave an $O(mn)$ algorithm for recognizing k-GC graphs. More specialized and efficient algorithms were given by Chang and Ho [5,4]. Let $S = v_1, \ldots, v_q$ be a q-clan of a k-GC graph G, i.e. $N(v_i)\backslash S = N(v_j)\backslash S$ for all i and j. A new k-GC graph G' can be formed by adding a new vertex v and edges (v, v_i) for $i = 1, \ldots, q$. To see that, simply perform a neighborhood test between the additional pairs of v and any vertex in $N(v_i) \setminus S$. All operations used here are surveyed in [14].

With these preparations, we will study the quality of the lower bound given in Theorem 4 by comparing it to an upper bound of $m(n, k)$ as stated below.

Lemma 3. *For any k-GC graph with $n/k \geq 2$, the size is bounded with $m(n, k) \leq nk - k^2$.*

Proof. Construct a graph G' with $|V(G')| = (n - n \bmod k)$. Evenly divide the vertices into exactly $p = |V(G')|/k \geq 2$ subsets $V_i (i = 1, \ldots, p)$, each having exactly k vertices. The edges $E(G')$ are only connecting vertices in two adjacent subsets, or $E(G') = \{(u, v) : u \in V_i, v \in V_{i+1},\ i = 1, \ldots, p-1\}$. Clearly G' is a k-GC graph. If $n \bmod k = 0$, since $|E(G')| = nk - k^2$, and $|V(G')| = n$, then $nk - k^2$ is a valid upper bound. Otherwise, $n \bmod k > 0$ vertices can be added to G'. Since V_1 is a k-clan of G', to construct a graph G of order n, first add a vertex v to G' by isosceles extension, then add the rest of the vertices by vertex cloning of v. The resultant graph G is a k-GC graph of order n, with $k(n \bmod k)$ new edges added to G', it is found that $|E(G)| = |E(G')| + k(n \bmod k) = nk - k^2$. This again gives the same upper bound of $m(n, k)$ as required. □

Some comments on the proof of the upper bound. First of all, the way to construct G' is not unique, an alternative way is presented here. First place edges between the vertex sets V_1, V_2 to form a complete bipartite graph $K(k, k)$. Then start adding other vertices by isosceles extension or vertex cloning freely. For example, after adding the first vertex v to V_1 by isosceles extension, repeat cloning v for all the remaining vertices, which will produce a $K(k, n-k)$. This graph has a diameter of 2, which is very different from the one constructed in the proof above, whose diameter is $\lceil n/k \rceil$. Starting from a bipartite graph $K(k, k)$, all diameter values in between can be constructed in many different topologies. Chang et al. [6] noticed this method in the construction of p-composition graphs.

It is well known that for many (n, k) pairs, there exists a family of k-GC graphs of order n with no more than $nk - k^2$ edges, e.g. Plesník described many samples in [14]. However, our emphasis here is a general upper bound for $m(n, k)$ with any (n, k) pairs satisfying $n/k \geq 2$, and we believe this is a new result. Plesník [14] had a conjecture, which says: *There exists a real constant c such that any k-GC graph of order $n \geq ck$ has size $m \geq kn - k^2$.* In light of this upper bound, Plesník's conjecture can be strengthened to $m(n, k) = kn - k^2$ under the same conditions. Another interesting observation is that any graph with a q-clan must have at least $q(n - q)$ edges, thus a graph can not be a minimum k-GC graph if there is a q-clan in it with $k < q < n - k$. Now we continue our study on the quality of the bounds:

Lemma 4. *For any k-GC graph with $k > 2$, and $\gamma = k/n < 1/2$,*

$$m(n, k)/l(n, k) < (1 - \gamma)(1 + 2\gamma) \text{ and} \tag{3}$$

$$m(n, k) - l(n, k) < k^2(1 - 2\gamma)/(1 + 2\gamma). \tag{4}$$

Proof. Fig. 1 clearly shows that

$$l(n, k) \geq \delta^* n/2 = \frac{kn}{1 + \frac{2k-2}{n-2}}.$$

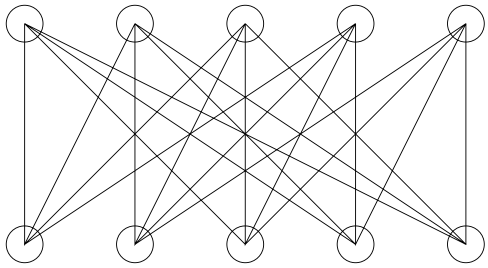

Fig. 2. A sample wreath $W(4, 1)$.

As $n > 2k$, it holds that $(2k-2)/(n-2) < 2k/n$, so

$$l(n,k) > \frac{kn}{1+2k/n}.$$

By Lemma 3, $m(n,k) \le nk - k^2$, thus

$$\frac{m(n,k)}{l(n,k)} < \frac{nk-k^2}{kn/(1+2k/n)} = (1-\gamma)(1+2\gamma).$$

The proof for $m(n,k) - l(n,k)$ follows similarly. □

The inequality (4) shows that the absolute difference $m(n,k) - l(n,k)$ is always no more than k^2, and it decreases as γ increases. As for the relative ratio, it is a quadratic function of γ that climaxes at 1/4 with the maximum of $1 + 1/8$, and symmetrically decreases as γ strays away from 1/4, until the minimum value of 1 is obtained at the two extremes when $\gamma \downarrow 0$ or $\gamma \uparrow 1/2$. Thus Lemma 4 indicates that when $\gamma \downarrow 0$ or $\gamma \uparrow 1/2$, our lower bound shall be very effective, and so is the upper bound given in Lemma 3.

In light of the quality of the bounds, some comments are in order. The derivation of these new bounds suggests that (1) the graphs constructed with such an upper bound come in a great deal of varieties, with different topologies to meet the requirements of different applications; (2) such graphs are flexible and easily extended. One can easily add new nodes to them upon system expansions and remain as k-GC graphs of sizes within the upper bound. Therefore, the constructed graphs in the proof of Lemma 3 are very useful since in the case of the Internet routing or wireless sensor network applications, the n is usually large relative to k. On the other hand, there may be much room for improvement when $\gamma = k/n$ is around 1/4, although the relative error is no more than 1/8 here. In the next section, we will see many instances that can actually obtain the lower bound with relatively large k, that is, when the absolute difference between the bounds is small.

4. Wreaths as minimum k-GC graphs

Wreaths are k-GC graphs of particular interest as the lower bound given in Theorem 4 can be reached by some of them, which then must be minimum k-GC graphs. They were also introduced in [14]. Wreaths are bipartite graphs. Let integers $s > 0$ and t satisfy $s \ge t \ge 0$, a wreath with parameters s and t, denoted as $W(s,t)$, can be defined as:

$$V(W(s,t)) = \{u_i, v_i : i \in [0 \dots s+t)\};$$
$$E(W(s,t)) = \{(u_i, v_j) : i \in [0 \dots s+t), j \in [i \dots i+t) \bmod s+t\}.$$

Here $[i \dots j)$ denotes the set of integers in interval $[i, j)$, and when the modulo operation is applied to a set of integers, the integers in the resultant set are the modulo of the integers in the operand set. The graph $W(s,t)$ is a bipartite, s-regular graph with order $n = 2(s+t)$ and size $m = s(s+t)$. Fig. 2 depicts a wreath of $W(4, 1)$, it is a 4-regular graph with order $n = 10$ and size $m = 20$. It is known to be a minimum 3-GC graph [3].

By Theorem 5, it becomes almost obvious that any wreath $W(s,t)$ is a k-GC graph with $k = s - t$: clearly, it is connected, and any two vertices within the same partite share at least $s - t$ common neighbors, and any two vertices from different partites have no neighbors in common. Now the question is, under what conditions, will some special wreaths reach the lower bound stated in Theorem 4, so that they must be minimum k-GC graphs? If wreath $W(s,t)$ achieves the lower bound, the following equations must hold, assuming $k > 2$:

$$sn/2 = \min(\lceil \delta^* \rceil n/2, k(n-2) - \lfloor \delta^* \rfloor (k-2)); \quad (5)$$

$$\delta^* n/2 = k(n-2) - \delta^*(k-2); \quad (6)$$

$$s + t = n/2; \quad (7)$$

$$s - t = k. \quad (8)$$

The $\min(\cdot)$ function in Eq. (5) can be simplified by showing that

$$\lceil\delta^*\rceil n/2 > k(n-2) - \lfloor\delta^*\rfloor(k-2) \quad (9)$$

can never happen. Should that happen, δ^* must be fractional to be consistent with Eq. (6). Also, Eq. (5) becomes

$$sn/2 = k(n-2) - \lfloor\delta^*\rfloor(k-2).$$

Substitute it into (9) to have $\lceil\delta^*\rceil n/2 > sn/2$. Since δ is fractional, thus $\lfloor\delta^*\rfloor < \delta^*$, from Fig. 1 it is clear that

$$sn/2 = k(n-2) - \lfloor\delta^*\rfloor(k-2) > \lfloor\delta^*\rfloor n/2.$$

Therefore $\lceil\delta^*\rceil > s > \lfloor\delta^*\rfloor$. This is absurd as s is an integer, thus (9) can never be true. So Eq. (5) can be replaced by:

$$\lceil\delta^*\rceil n/2 \le k(n-2) - \lfloor\delta^*\rfloor(k-2); \quad (10)$$

$$sn/2 = \lceil\delta^*\rceil n/2. \quad (11)$$

Subtracting (6) from (10) yields

$$(\lceil\delta^*\rceil - \delta^*)n/2 \le (\delta^* - \lfloor\delta^*\rfloor)(k-2). \quad (12)$$

With (7) and (8), rewrite (6) as

$$\delta^* = s - \frac{t(t-1)}{s-1}. \quad (13)$$

To analyze (12), two cases are looked into. In the case that δ^* is an integer, (12) is automatically satisfied. Further, from (11) it is found that $s = \lceil\delta^*\rceil = \delta^*$, thus $t(t-1) = 0$ is derived from (13).

Now consider the case when δ^* is fractional. Let $\{x\} = x - \lfloor x\rfloor$ denote the fractional part of x. And with (7) and (8), rewrite (12) as:

$$(1-\{\delta^*\})n/2 \le \{\delta^*\}(k-2) \Longrightarrow 1 - \{\delta^*\} \le \frac{k/2-1}{s-1}. \quad (14)$$

By Eq. (13), and equation $s = \lceil\delta^*\rceil$ from (11):

$$1 - \{\delta^*\} = \lceil\delta^*\rceil - \delta^* = \frac{t(t-1)}{s-1}.$$

Substitute this into (14) to get

$$\frac{t(t-1)}{s-1} \le \frac{k/2-1}{s-1}.$$

This leads to the following theorem:

Theorem 6. *Wreath $W(k+t, t)$ is a minimum k-GC graph with $n = 2k + 4t$ when $k > 2$ and $t(t-1) \le k/2 - 1$.*

When $t = 0$, the resultant graph is simply a complete bipartite graph $K(k, k)$, which has already been discussed thoroughly in the literature. By Theorem 6, $W(4, 1)$ is a minimum 3-GC graph with $n = 10$ and $m = 20$, which is the most special case for all 3-GC graphs as noted by Bosíková [3]. According to her result, all other 3-GC graphs with $n \ge 2k$ stay at the upper bound of $nk - k^2$. Because of the condition $n = 2k + 4t$, Theorem 6 does not cover the (n, k) pairs satisfying $n - 2k \bmod 4 \ne 0$, which is an interesting direction for further investigation. Recall that k-GC graphs for cases like $n = 2k + 4t + i$ can be constructed from the minimum wreaths $W(k + t, t)$ by arbitrarily cloning i new vertices. As long as $i \le k - 2t$, the new k-GC graphs will have sizes no more than the bound $nk - k^2$. But there is still an imminent question: are the wreaths given in Theorem 6 the only minimum k-GC graphs for the corresponding (n, k) pairs? We address this problem in the next section.

5. Beyond the minimum wreaths

Now we know that when $k > 2$, $t(t-1) \le k/2 - 1$, wreath $W(k + t, t)$ is a minimum k-GC graph with $n = 2k + 4t$ and $m = (k + 2t)(k + t)$. Is this the only minimum k-GC graph of order n? If not, what characteristics would describe all such minimum k-GCs? We first explore some of the characteristics of minimum k-GC graphs.

Theorem 7. *For any minimum k-GC graph of order n with $n/k \ge 2$, its diameter d is bounded by*

$$d \le n/k + 1 - 2/k, \quad (15)$$

with $<$ if $m(n, k) < nk - k^2$.

Proof. Consider the distance decomposition of the graph for any vertex u, let r be the eccentricity of u. There would be exactly $\deg(u) \geq k$ edges between D_0^u and D_1^u. For each $i = 1, \ldots, r-1$ and any vertex $v \in D_{i+1}^u$, since there are at least k internally disjoint $u - v$ geodesics, there are at least k edges from v to D_i^u. Hence there are at least $k|D_{i+1}^u|$ edges between D_{i+1}^u and D_i^u, and $|D_i^u| \geq k$. Summing all those edges up, it gives a lower bound of $m(n, k)$:

$$m(n,k) \geq k + \sum_{i=2}^{r} |D_i^u|k \geq (r-2)k^2 + 2k.$$

Recall the upper bound of $nk - k^2$ on $m(n, k)$ in Lemma 3, and it is clear that $nk - k^2 \geq (r-2)k^2 + 2k$, which yields

$$r \leq n/k + 1 - 2/k,$$
$$\text{with } < \text{ if } m(n,k) < nk - k^2.$$

As u is chosen arbitrarily, that is also the upper bound of the maximum eccentricity, which is the graph diameter by definition. □

Take a look at a minimum k-GC graph of size $n = 2k > 4$. By Theorem 7, the diameter $d \leq 2$. Then by Theorem 2, $K(k, k)$ is the only minimum k-GC graph. Obviously, $K(k, k)$ is k-regular, equally bipartite, with a uniform eccentricity over all vertices. In the context of Theorem 6, $K(k, k)$ corresponds to the case of $t = 0$, we naturally wonder if those properties remain for $t > 0$. The following lemmas address this question.

Lemma 5. *If $n = 2k + 4t$, $k > 2$ and $0 < 4t \leq k + 2$, in a minimum k-GC graph G of order n, any vertex of degree δ has an eccentricity of $r = 3$.*

Proof. Observe that the size of a k-GC graph $W(k + t, t)$ is $m = nk - k^2 - (k - 2t)t$, and $m < nk - k^2$ as $(k - 2t)t > 0$ can be readily derived from $k > 2$ and $0 < 4t \leq k + 2$. Therefore $m(n, k) \leq m < nk - k^2$, and by Theorem 7 the diameter of such a minimum k-GC graph $d < 1 + (2k + 4t - 2)/k \leq 4$. For minimum k-GC graphs $\delta \geq k$, and $\delta n/2 \leq m(n, k) \leq m$, which gives $k \leq \delta \leq k + t$. Consider the distance decomposition of G from a vertex u with $\deg(u) = \delta$. Clearly $|D_1^u| = \delta$, and

$$|D_0^u| + |D_1^u| = 1 + \delta \leq 1 + k + t < n,$$

thus $r \geq 2$. Suppose $r = 2$. Clearly $|D_r^u| = n - 1 - \delta$, from (1) in Lemma 1:

$$\begin{aligned} m(n,k) &\geq kn - k^2 - (\delta - k)(k - 1 - (n - 1 - \delta)/2) \\ &= k(n-k) - (\delta - k)(\delta + 2k - n - 1)/2 \\ &= k(n-k) - (\delta - k)(\delta - 4t - 1)/2 \\ &\geq k(n-k) - \max_{k \leq \delta \leq k+t} (\delta - k)(\delta - 4t - 1)/2 \\ &= k(n-k) - \max(0, t(k - 3t - 1)/2). \end{aligned}$$

When $k \leq 3t + 1$, it reduces to $m(n, k) \geq k(n - k)$, which contradicts $nk - k^2 > m(n, k)$! When $k > 3t + 1$, it reduces to $m(n, k) \geq k(n - k) - t(k - 3t - 1)/2$.

$$\begin{aligned} m(n,k) - m &\geq k(n-k) - t(k - 3t - 1)/2 - m \\ &= kt - 2t^2 - t(k - 3t - 1)/2 \\ &= t(k - t + 1)/2 > 0. \end{aligned}$$

Thus $m(n, k) > m$, which contradicts $m \geq m(n, k)$! So $r \neq 2$ and since $2 \leq r \leq d < 4$, it must be true that $r = 3$. □

When $t = 0$, the minimum graph is k-regular, this lemma below states that regularity remains even when $t > 0$. The proof for this lemma is unfortunately quite lengthy.

Lemma 6. *If $n = 2k + 4t$, $k > 2$ and $t(t - 1) \leq k/2 - 1$, any minimum k-GC graph G of order n is $(k + t)$-regular.*

Proof. The cases of $t = 0$ is already known, so $t > 0$ is assumed hereafter. By Theorem 6, $m(n, k) = (k + t)n/2$, so $k + t$ is the average degree and it suffices to show $\delta = k + t$. Let $t' = \delta - k \geq 0$, and $t' \leq t$ as $\delta \leq k + t$. Choose a vertex u such that $\deg(u) = \delta$. By Lemma 5, the eccentricity of u is $r = 3$. As in the proof of Theorem 7, there are δ edges between D_0^u and D_1^u. And for $i = 1, \ldots, r - 1$, there are at least $k|D_{i+1}^u|$ edges between D_{i+1}^u and D_i^u. Count those edges to get $m' = (n - 1 - \delta)k + \delta = k(n - k) + t' - t'k$. The number of uncounted edges is $m(n, k) - m' = t'(k - 1) - t(k - 2t)$. Now consider the uncounted edges that have at least one end in D_r^u, and let q denote their number. For each vertex in D_r^u, the degrees associated with those q edges count at least $t' = \delta - k$. These degrees may be associated with edges to D_{r-1}^u or within D_r^u. If $|D_r^u| \geq 2$, then $q \geq |D_r^u|t'/2 \geq t'$. But if $|D_r^u| = 1$, then those uncounted edges must have the other end in D_{r-1}^u, and clearly $q \geq t'$. So, $m(n, k) - m' \geq q \geq t'$, which simplifies to $t' \geq t(k - 2t)/(k - 2)$. Since $t(t - 1) \leq k/2 - 1$, it is found that $t(k - 2t)/(k - 2) \geq t - 1$, with equality obtained if and only if $t(t - 1) = k/2 - 1$. Therefore, $t' \geq t - 1$.

Suppose $t' = t - 1$, then $t(k-2t)/(k-2) = t - 1$ and $t(t-1) = k/2 - 1$. And $m(n,k) - m' = t'(k-1) - t(k-2t) = (t-1)(k-1) - (t-1)(k-2) = t - 1$. From $m(n,k) - m' \geq q \geq t' = t - 1$, clearly $m(n,k) - m' = q = t'$, so all uncounted edges are those q edges associated with D_r^u. Therefore D_1^u and D_2^u must both be independent sets. Since $k > 2$ and $t(t-1) = k/2 - 1$ implies $t \geq 2$, then $m(n,k) - m' = t - 1 \geq |D_r^u|(t-1)/2$ implies $|D_r^u| \leq 2$.

a. $|D_r^u| = 1$. Let $D_r^u = \{v\}$, and clearly $\deg(v) = k + q = \delta$. As $|D_2^u| = n - 1 - \delta - 1$, there are $|D_2^u| - \delta = 2t$ vertices in D_2^u not directly connected to v. Since D_2^u is independent, and $N(v) \subset D_2^u$, they must have distances greater than 2 from v. As Lemma 5 requires $r = 3$ for v, then $|D_r^v| \geq 2t \geq 4$. But since $\deg(v) = \delta$, we must also have $|D_r^v| \leq 2$ as $|D_r^u|$. A contradiction!
b. $|D_r^u| = 2$. For any $v \in D_r^u$, if $\deg(v) > \delta$, total degrees associated with uncounted edges will be greater than $2t'$, which contradicts $q = t'$, therefore $\deg(v) = \delta$. Also by $q = t'$, all the uncounted edges must stay within D_r^u. So v has exactly k edges going to D_2^u, and there are exactly $|D_2^u| - k = n - 1 - \delta - 2 - k = 3t - 2 \geq 4$ vertices not directly connected to v, which leads to the same contradiction as in case a.

Thus $t' = t$, $\delta = k + t$, which is the average degree, so G is $(k+t)$-regular. □

Now since each vertex has the same degree, each vertex has the minimum degree. From Lemma 6 we have

Corollary 2. *If $n = 2k + 4t$, $k > 2$ and $t(t-1) \leq k/2 - 1$, any minimum k-GC graph G of order n has a uniform eccentricity over all vertices.*

Furthermore, we will show that just like the case when $t = 0$, those special minimum k-GC graphs are also bipartite, and each partite has the same order. We thus have:

Theorem 8. *If $n = 2k + 4t$, $k > 2$ and $t(t-1) \leq k/2 - 1$, any minimum k-GC graph G of order n is an equally bipartite, $(k+t)$-regular graph, and vice versa.*

Proof. Since the case of $t = 0$ is already known, assume $t > 0$ hereafter. Consider the distance decomposition of G with regard to $u \in V(G)$. To be concise, let $n_i^u = |D_i^u|$, and $e_i^u = |\{(x,y) \in E(G) : x, y \in D_i^u\}|$. Clearly, $n_0^u = 1$, $n_1^u = \delta$. By Lemma 6, each vertex has the same degree of $\delta = k + t$, and by Lemma 5, the same eccentricity $r = 3$.

Consider D_2^u, by Theorem 5, each vertex in it has at least k edges to D_1^u. Thus D_2^u has at least $n_i^u k$ edges to D_1^u. And by the same reason, D_3^u has at least $n_3^u k$ edges to D_2^u. Since D_i^u has $n_i^u \delta$ degrees in total, thus

$$n_2^u \delta \geq n_2^u k + n_3^u k + 2e_2^u. \tag{16}$$

Similar arguments for D_1^u and D_3^u will give

$$n_1^u \delta \geq n_1^u + n_2^u k + 2e_1^u, \tag{17}$$

$$n_3^u \delta \geq n_3^u k + 2e_3^u. \tag{18}$$

With $n_1^u = \delta = k + t$, and $e_i^u \geq 0$, the inequality of (17) comes to $k + 2t - 1 + t(t-1)/k \geq n_2^u$. As $0 < t(t-1) \leq k/2 - 1$, which implies $0 \leq t(t-1)/k < 1/2$, it leads to $n_2^u \leq k + 2t - 1$. As $n_0^u + n_1^u + n_2^u + n_3^u = n$, and $n_0^u = 1$, $n_1^u = k + t$, then $n_3^u = k + 3t - 1 - n_2^u \geq t$. Substitute $n_2^u = k + 3t - 1 - n_3^u$ into $n_2^u t \geq n_3^u k$, which comes from inequality (16) by ignoring e_2^u, and simplify to get $n_3^u \leq t + t(2t-1)/(k+t)$. By $t(t-1) \leq k/2 - 1$, which implies $t(2t-1) < k + t$, it leads to $n_3^u \leq t$ as n_3^u is an integer. Therefore $n_3^u = t$, and so $n_2^u = k + 2t - 1$. Substituting the values of all n_i^u's into Eqs. (16)–(18), to have the following results:

$$e_1^u \leq t(t-1)/2,$$
$$e_2^u \leq t(2t-1)/2,$$
$$e_3^u \leq t(t-1)/2.$$

Select a vertex $v \in D_3^u$ and $w \in D_2^u \setminus N(v)$, so $d(w,v) > 1$. If $d(w,v) = 2$, then $|N(w) \cap N(v)| \geq k$, by Theorem 5. However,

$$N(v) \subset D_3^u \cup D_2^u \Longrightarrow N(w) \cap N(v) \subset D_3^u \cup D_2^u.$$

Then all the internally disjoint $(v-w)$-geodesics are within $D_3^u \cup D_2^u$, with each geodesic containing a distinct edge within either D_2^u or D_3^u (see those dashed edges from v and w in Fig. 3).

Thus $e_2^u + e_3^u \geq k$, but $e_2^u + e_3^u \leq t(t-1)/2 + t(2t-1)/2 = 3t^2/2 - t < k$. A contradiction! So $d(w,v) = 2$ is impossible, then $d(w,v) = 3$ must be the case as $r = 3$, and thus

$$D_2^u \setminus N(v) \subseteq D_3^v.$$

As $u \in D_3^v$, then D_3^v is a strict superset, so

$$n_2^u - |D_2^u \cap N(v)| \leq n_3^v - 1,$$

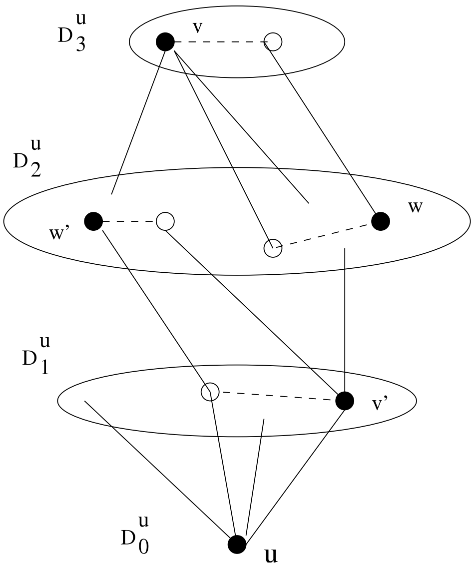

Fig. 3. Dashed edges to enable $d(w, v) = 2$ or $d(w', v') = 2$.

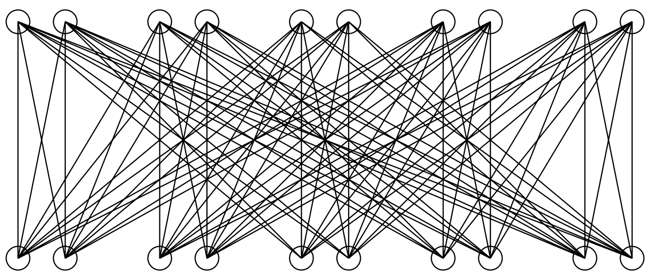

Fig. 4. A minimum 6-GC graph other than $W(8, 2)$.

which simplifies to $|D_2^u \cap N(v)| \geq k+t$, since $n_3^v = n_3^u$ as u is arbitrarily chosen. As $|N(v)| = k+t$, it must be that $N(v) \subset D_2^u$, thus v has no edges in D_3^u, so D_3^u must be an independent set as v is arbitrarily chosen.

Similarly, D_1^u is also an independent set (see those dashed edges from v' and w' in Fig. 3). Since there are no edges between D_1^u and D_3^u, $D_1^u \cup D_3^u$ is an independent set as well. Further, D_1^u must have $(k+t)^2 - (k+t)$ edges to D_2^u, and D_3^u must have $t(k+t)$ edges to D_2^u. So D_2^u has $(k+t)^2 - (k+t) + t(k+t)$ incoming edges, which simplifies to $(k+t)(k+2t-1)$. But that is exactly $n_2^u\delta$, the total degree it can have! So D_2^u must also be independent. Thus G has one partite of $D_0^u \cup D_2^u$, and the other of $D_1^u \cup D_3^u$, with each partite having exactly the same number of $k+2t$ vertices.

On the other hand, if $n = 2k + 4t$, $k > 2$ and $t(t-1) \leq k/2 - 1$, any equally bipartite graph G of order n that is $(k+t)$-regular, will pass the neighborhood test, thus it is a k-GC graph by Theorem 5. Also, its size is the minimum size by Theorem 6, so G is a minimum k-GC. □

For example, when $k = 6$, $t = 2$, besides $W(8, 2)$ as a minimum 8-GC graph of order 20, we also have the graph shown in Fig. 4, which is not a wreath, but a 2-composition of $W(4, 1)$.

This example clearly shows that wreaths generally are not the only solution. But when $t = 0, 1$, wreaths become the only graphs that satisfy all the conditions, thus they become the only solutions.

6. Conclusion

With the increasing complexity of computer systems, robustness has been taken into account to the system design with the minimum cost. In this paper, we have studied k-geodetically connected graphs with the minimum number of edges, which is important to the cost effective robustness design in many computer communication systems. We have developed a tighter lower bound of $m(n, k)$ as well as a good upper bound and shown how to construct a great variety of k-GC graphs that stay within the upper bound. Vertices can be easily added to such a graph to derive a new k-GC graph that still stay within the upper bound. In addition, we have completely determined all the minimum k-GC graphs n, if $n = 2k+4t$, $t(t-1) \leq k/2-1$ and shown that they are highly regular: uniform degree, uniform eccentricity, and equally bipartite.

However, the limit of our current work is that it makes n jumps by four. When $n = 2k + 4t + i$ for $i = 1, 2, 3$, our best conjecture is that we can carry out vertex cloning on the minimum k-GC graph of $n = 2k + 4t$ by i times to obtain a k-GC graph of $n = 2k + 4t + i$, but the minimality cannot be guaranteed yet. For example, for $(n, k) = (12, 3)$, we have found the minimum k-GC has $m(12, 3) = 27$ edges. But the k-GC obtained via vertex-cloning from minimum 3-GC $W(4, 1)$ with size $n = 10$, is $20 + 4 * 2 = 28 > 27$.

Another promising approach is to develop better lower bounds on $m(n, k)$, which may help adventure into more unknown (n, k) pairs. Let $M(n, k)$ denote the set of all the minimum k-GC graphs of order n. For the special cases when $n = 2k + 4t$, $t(t - 1) \leq k/2 - 1$, we have $m(n, k) = (k + t)(k + 2t)$ and $M(n, k)$ contains all such graphs as described in Theorem 8. Let $\delta(n, k) = \min\{\delta(G) : G \in M(n, k)\}$. A k-GC of order $n+1$ can be constructed via vertex-cloning of that vertex whose degree is $\delta(n, k)$, so we have $m(n + 1, k) \leq m(n, k) + \delta(n, k)$, which gives an upper bound for $m(n + 1, k)$. Those inequalities can be used to get new bounds for $m(n, k)$, especially when a good approximation of $\delta(n, k)$ can be obtained. For example, the δ^* given in Theorem 4 could serve this purpose, which could be a good starting point for future research.

Acknowledgements

We would like to express our gratitude to the anonymous referees for their comments towards improving this paper. The first author Yingjie Lan would also like to acknowledge the financial aid for this research from Project 71072050 supported by NSFC.

References

[1] J.G. Apostolopoulos, T. Wong, W. Tan, S.J. Wee, On multiple description streaming with content delivery networks, in: IEEE INFOCOM, June 2002.
[2] B. Bollobás, S. Eldridge, On graphs with diameter 2, J. Combin. Theory Ser. B 21 (1976) 17–33.
[3] M. Bosíková, Minimum 3-geodetically connected graphs, Discrete Appl. Math. 129 (2003) 263–283.
[4] J.-M. Chang, C.-W. Ho, The recognition of geodetically connected graphs, Inform. Process. Lett. 65 (1998) 81–88.
[5] J.-M. Chang, C.-W. Ho, Recognizing hinge-free line graphs and total graphs, Taiwanese J. Math. 5 (2001) 789–801.
[6] J.-M. Chang, C.-W. Ho, C.C. Hsu, Y.L. Wang, The characterizations of hinge-free networks, in; Proc. Int. Comput. Symp. Algorithms, Taiwan, 1996, pp. 105–112.
[7] G. Chartrand, L. Lesniak, Graphs and Digraphs, 3rd ed., Chapman and Hall, London, 1996.
[8] R.E. Entringer, D.E. Jackson, P.J. Slater, Geodetic connectivity of graphs, IEEE Trans. Circuits Syst. CAS-24 (1977) 460–463.
[9] A.M. Farley, A. Proskurowski, Self-repairing networks, Parallel Process. Lett. 3 (1993) 381–391.
[10] A.M. Farley, A. Proskurowski, Minimum self-repairing graphs, Graphs Combin. 13 (1997) 345–351.
[11] D. Ganesan, R. Govindan, S. Shenkar, D. Estrin, Highly-resilient, energy-efficient multipath routing in wireless sensor networks, Mobile Comput. Commun. Rev. 1 (2) (1969).
[12] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[13] D.E. Jackson, R.E. Entringer, Minimum k-geodetically connected graphs, Congr. Numer. 36 (1982) 303–309.
[14] J. Plesník, Towards minimum k-geodetically connected graphs, Networks 41 (2) (2003) 73–82.
[15] J. Plesník, Minimum k-geodetically connected digraphs, Networks 44 (2004) 243–253.
[16] F. Wang, L. Gao, J. Wang, J. Qiu, On understanding of transient interdomain routing failures, in: Proceedings of 13th IEEE International Conference on Network Protocols, ICNP, Boston, MA, November 2005.